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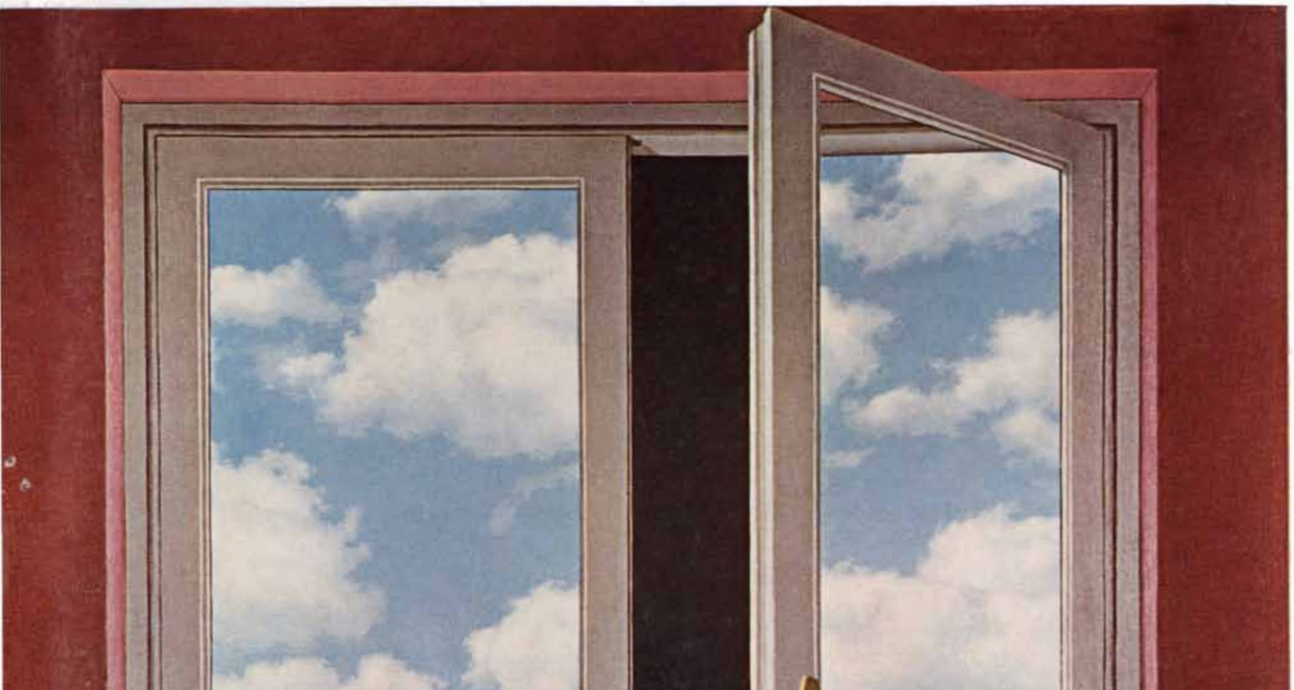
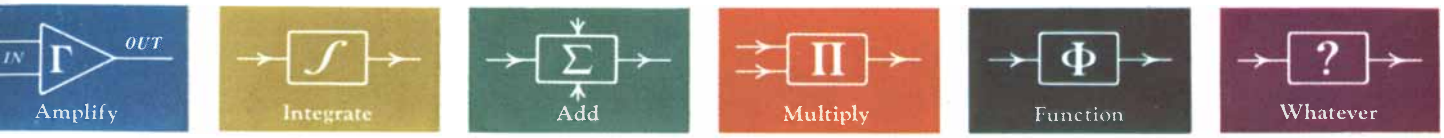


MATHEMATICS IN THE MODERN WORLD

SIXTY CENTS

September 1964



AN ALPHABET of ANALOGICAL APPLICATIONS

A is for Amplifying, accurately and articulately, according to algebraic or analytic authority, automatically to achieve aspirations. *B* is for Balancing bridges and other networks, nonlinear as well as linear, in order to compute or test or control something. *C* is for Computing, controlling, causally reversing, closing of many categories of loops, in cosmology, communications, chemistry, et cetera. *D* is for Differentiating of variables, for example with respect to time; also dividing one by another; also dynamically transforming them. *E* is for Educating one's self or others, using electronic models of the objects in point: quick to assemble, to operate, and to alter. *F* is for Finding the finest set of design parameters for, say, a feedback system, giving maximum performance and reliability. *G* is for Generating functions in great variety, whether of time or of other variables, giving glorious graphical generality. *H* is for Hybridizing of logical and analogical apparatus, thus harmonizing the continuous and the discrete by natural procedures. *I* is for Inversion in the sense of sign-reversal, but more broadly too; also integrating; also isolation of influences; also inciting intuitions. *J* is for Judging and judicating, perhaps automatically, in the course of evaluating results against justifiable criteria. *K* is for Keeping a physical state intact or predictable, in spite of disturbances, and thus assuring some static or dynamic outcome (desirable). *L* is for Linearizing the nonlinear, if desired; conversely, for example, limiting or otherwise liberalizing the linear realm. *M* is for Modelling, measuring, and manipulating, as on the front cover, but also multiplying, memorizing, and maximizing or minimizing. *N* is for Neutralizing or nullifying of nuisances or unwanted effects in an experiment or on-line process; also normalizing. *O* is for Operating and organising systems, including self-organising ones; also optimizing by parameter search; also oscillating. *P* is for Perceiving physical states, or patterns; also predicting and prolonging desired conditions; also pedagogy on dynamical theory and practice. *Q* is for Questioning Nature, sometimes obliquely, always diplomatically, flattering her through imitation; also quickening of responses. *R* is for Realizing in the routine sense, and also in the sense of making real, as when making mathematics come to life. *S* is for Simulating or synthesizing of systems, whether already on hand or simply on the dwg. boards; also statistics and stochastics. *T* is for Testing of machines and components, as well as of brainwaves and brainstorms; also training of technologists; also trying out of tolerances. *U* is for Unilateralizing, or restricting causality to a single direction when this is required, between subsystems; also, unlearning the untrue. *V* is for Varying experimental conditions automatically, according to space-filling excursions, regular or random; also vibrating. *W* is for Weighting of averages, generalizing the simple summation of functions of time; also waveform development and analysis. *X* is for X-ray instrumentation in experimental physics, as for the velocity regulator of the Mössbauer apparatus. *Y* is for Yield-estimation for planned or existing production processes, in which intricate or implicit inter-relationships obtain. *Z* is for Zero-seeking, both in the sense of nulling and in the finding of the roots of a function. *&* itself is for all additional applications of analogical artifices.

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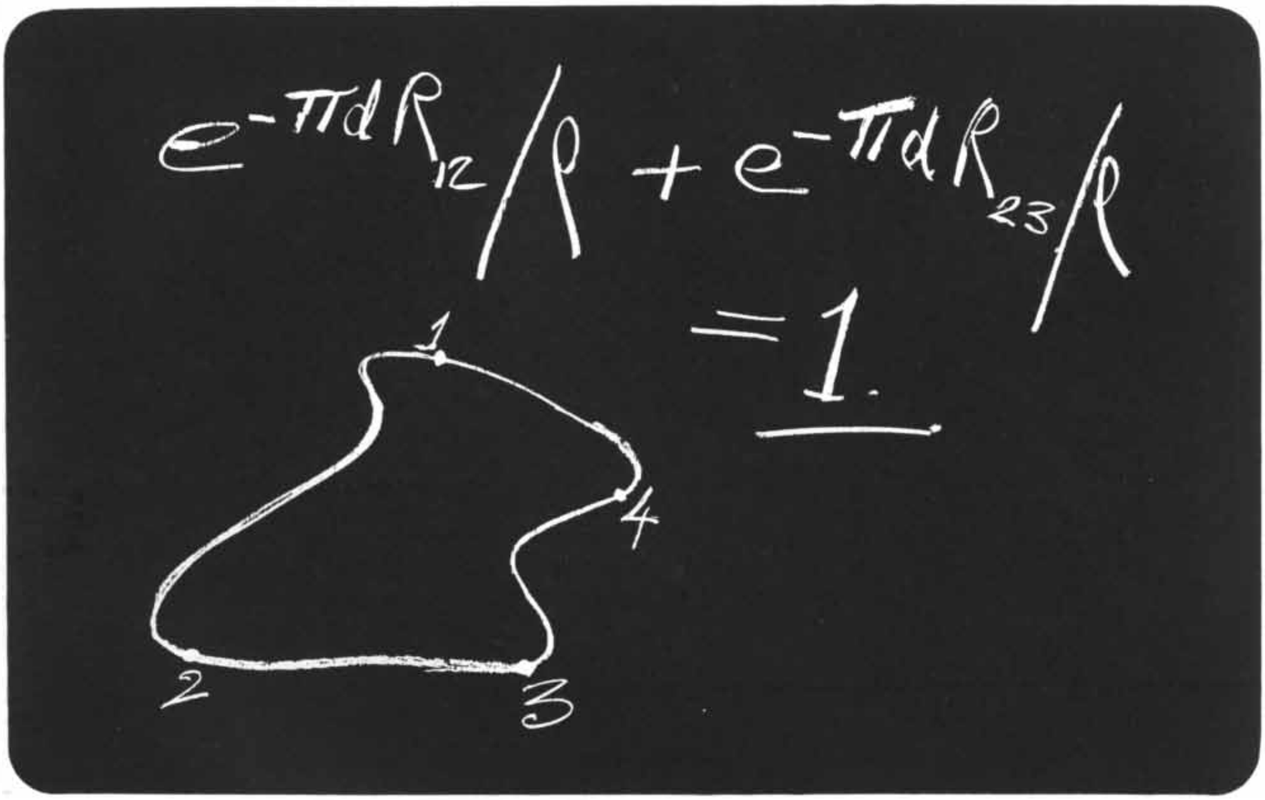


Ohm's Law - 1964

It's called van der Pauw's theorem. And in my opinion, it is a most striking example of how a rather sophisticated piece of mathematics led to an extremely practical result. Why? Because of its outstanding elegance and simplicity. Before van der Pauw came along, the determination of specific resistivity - especially in the case of semi-conductors - while not exactly difficult, was a time-consuming and lengthy procedure. The best procedure available was to grind cylindrical or prismatic rods and to determine accurately both length and cross section.

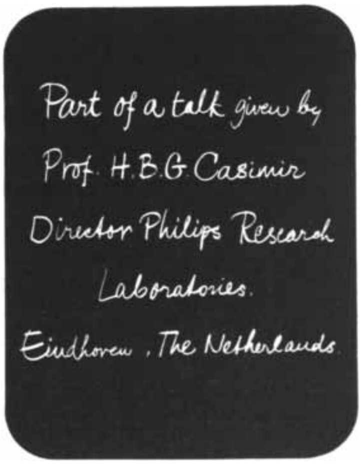
Van der Pauw, on the other hand, showed that it is possible to measure the specific resistivity of a flat sample of constant thickness but otherwise arbitrary shape merely by placing four small contacts (1, 2, 3, 4) on the circumference. Then if we define R_{12} as the ratio of the voltage between 1 and 2, and the current between 3 and 4, and similarly R_{23}, we have:

$$e^{-\pi d R_{12}/\varrho} + e^{-\pi d R_{23}/\varrho} = 1\,.$$

where d is the thickness of the sample.

Let me indicate the proof. First take an infinite half plane. Then all potentials are logarithmic and the proof is elementary. Next map conformly to obtain the desired contour and remember that resistance is a conformal invariant. That's all. Very simple once you begin to think about it - which apparently no one did before van der Pauw. And very useful; one measures two resistance values and reads ϱ/d from a set of curves. We not only apply this in the research laboratories, but also in our factories to test specific resistance.

Incidentally, if you would like the complete proof of van der Pauw's work, send for Philips Technical Review vol. 13, No. 1. It's called "A Method of measuring specific resistivity and Hall Effect of discs of arbitrary shape". I think you'll enjoy it.



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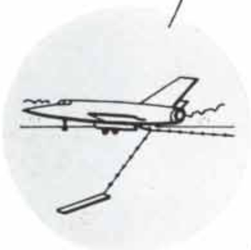
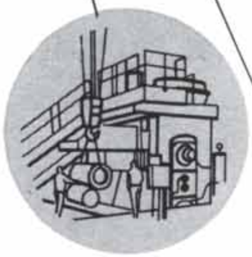
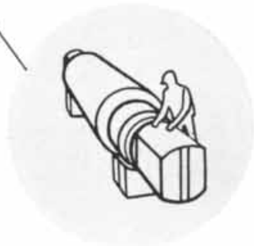


West Germany: A Bliss arresting gear "catches" a landing U. S. A. F. jet fighter on the runway after a brake failure.

Brazil: Auto body sections are stamped out on huge Bliss mechanical presses at a plant in São Paulo.



Australia: Vitally-needed aluminum strip comes whirling off a new Bliss cold mill near Sydney.



Chicago: A Bliss Mackintosh-Hemphill roll is readied for service in a rolling mill at a giant steel plant.

Memphis: Antiquated fire alarms are replaced by a modern, more extensive Bliss-Gamewell coded alarm system.

Cape Kennedy: Bliss-built nozzles and engine parts help guide a Minuteman missile 5,000 miles down range.

Tampa: Revolutionary Bliss-Eagle traffic control system uses an electronic brain to synchronize traffic flow.

Even as these events are taking place, others are in the making. Bliss dedicates a new press-building plant for Israel . . . ships a complete can-making machinery line to a big food packer . . . announces a new plant in Canada to manufacture precision timers and other products . . . negotiates a contract for prototype production of parts for an advanced rocket system. Everywhere in this growing, changing world, you'll find Bliss . . . growing and changing with it. Write for our 20-page booklet of Bliss activities.

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Established 1845 SCIENTIFIC AMERICAN September 1964 Volume 211 Number 3

ARTICLES

DEPARTMENTS

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THE COVER

The cover of this issue poetically represents the theme of the issue: mathematics in the modern world. It is a detail of a painting by René Magritte; the entire painting is shown in the small reproduction at left. The painting symbolizes that aspect of mathematics which makes outrageous new assumptions to erect new mathematical systems. Through the closed half of the window one sees a sunlit sea and sky. The open half shows that behind the window is nothingness; one assumes, then, that the scene is painted on the glass. But wait! At the top of the open half of the window one can see the window frame through the glass.

THE ILLUSTRATIONS

Cover painting by René Magritte

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RECENT findings

RESEARCH LABORATORIES

Ford

MOTOR COMPANY

GIANT QUANTUM EFFECT IN SUPERCONDUCTORS

Ford Motor Company scientists report a macroscopic system that obeys the laws postulated by the Quantum Theory

Superconductivity—the complete disappearance of resistance to electrical current in certain metals at temperatures near absolute zero—has long tantalized scientists with unfulfilled promises of technological advances in computers and communication. The principal obstacle to future advances was a lack of basic understanding of the nature of superconductivity. Now Ford scientists report confirmation of a novel concept that basic laws governing the current flow in superconductors are identical with the laws, postulated by the Quantum Theory, describing the motion of electrons in the atom themselves.

These laws are an attempt to account for atomistic nature in terms of wave characteristics presumed associated with particles. This allows a strange kind of addition—two large waves can combine "out of phase"—the crests of one joining the troughs of another—to produce a wave disturbance *smaller* than the original waves.

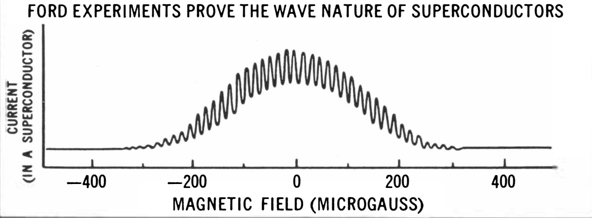
ANALOGY BETWEEN ELECTRON ORBITS IN ATOMS AND ELECTRON CURRENT IN SUPERCONDUCTORS

An electron traveling in its orbit is a lossless current comparable to the lossless current observed in superconductors. If this analogy is valid, other quantum effects proposed for atoms should be applicable to superconductors. For example: wave properties in phase add constructively; waves out of phase cancel.

Wave-like characteristics have been discovered in supercurrents equivalent to those hypothesized by the Quantum Theory as existing in atomic orbits. However, in the Ford experiments the "orbits" extend over several inches, distances hundreds of millions of times larger than those of the atomic world.

This giant quantum effect was detected by measuring the flow of electrical current between two superconductors. The wave properties of the electron current between the two superconductors allow current to combine with varying phases. When they are in phase, the resulting current is larger than either of the original currents; when they are out of phase, the resulting current is smaller. Whether the interference is in or out of phase depends upon the strength of the magnetic flux within the superconducting material.

FORD EXPERIMENTS PROVE THE WAVE NATURE OF SUPERCONDUCTORS



By watching the current strength change on conventional measuring devices, Ford scientists now can "see" the giant quantum wave motion of the electrons.

Ford scientists view this breakthrough in the control of superconductivity as possibly leading to a whole new family of solid state devices. For instance, a superconducting magnetometer has been built in the Ford Laboratories with a field resolution of small fractions of a microgauss, the highest resolution yet obtained in the measurement of magnetic flux. Amplifiers, oscillators and computer elements employing superconducting quantum effects are other applications being explored at the Ford Research Laboratories.

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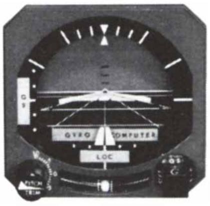
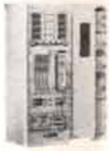
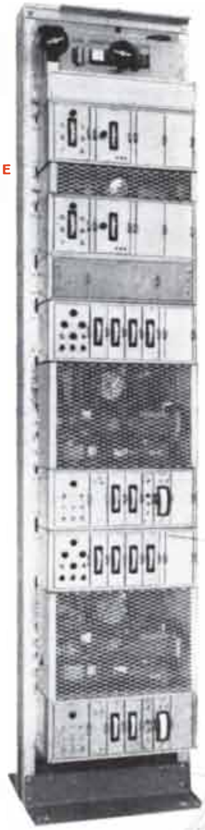


FLIGHT DIRECTOR

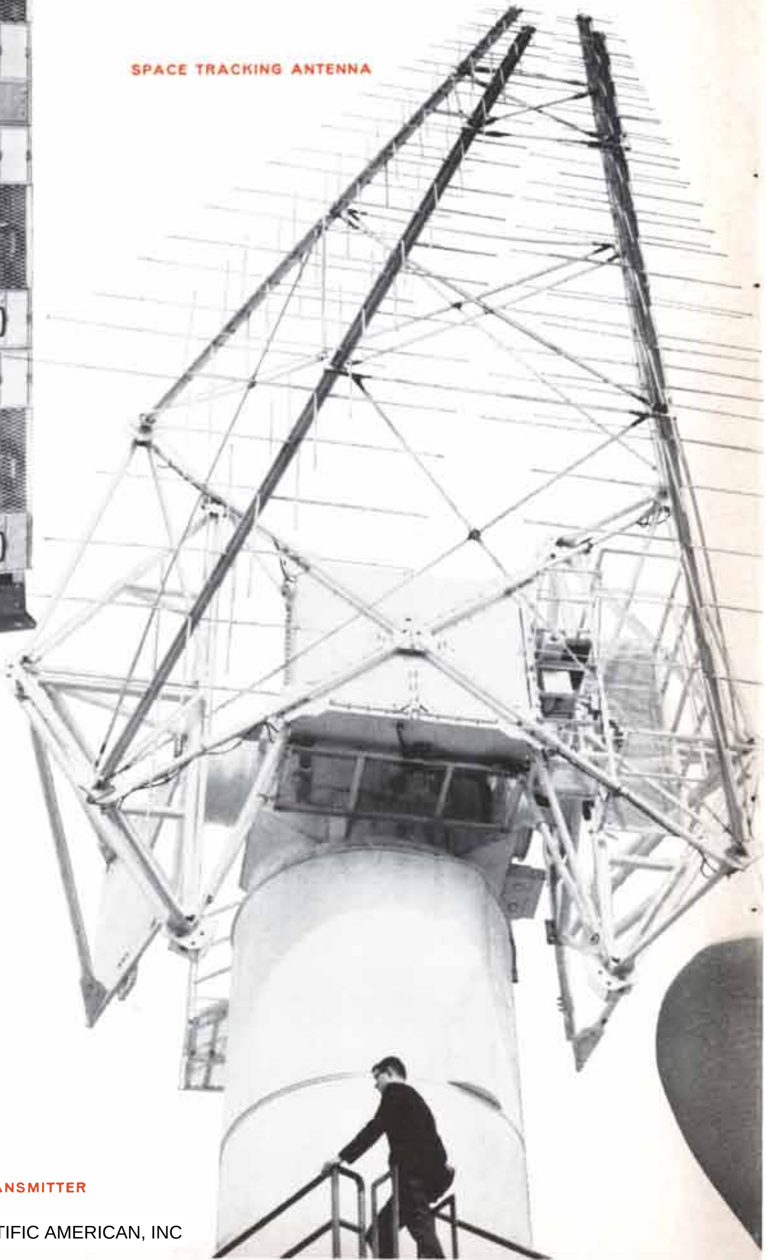
MECHANICAL FILTER



CONTROL CONSOLE



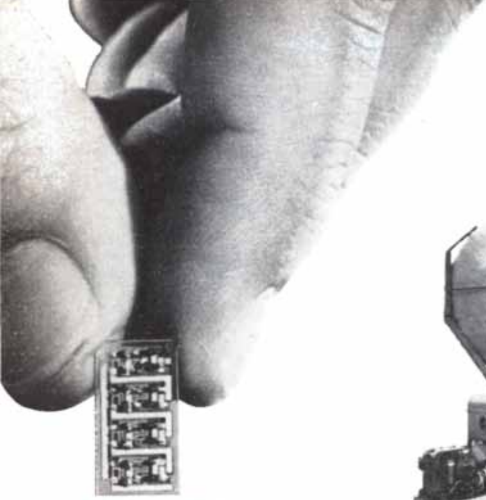
MICROWAVE



SPACE TRACKING ANTENNA

PORTABLE TACTICAL TRANSCEIVER

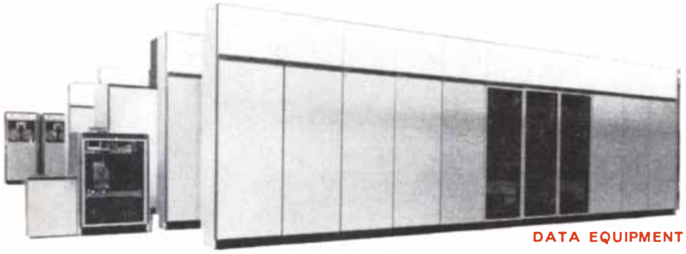
FM BROADCAST TRANSMITTER



MICROMINIATURE CIRCUITRY

TRANSPORTABLE TACTICAL COMMUNICATION STATION

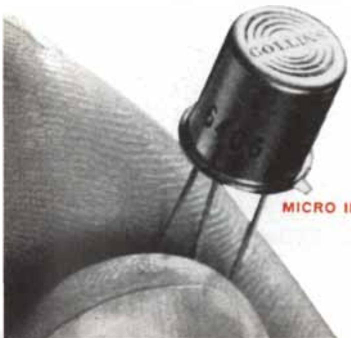
TACTICAL DATA SYSTEM



DATA EQUIPMENT

AIRBORNE TRANSCEIVER

AIRBORNE WEATHER RADAR



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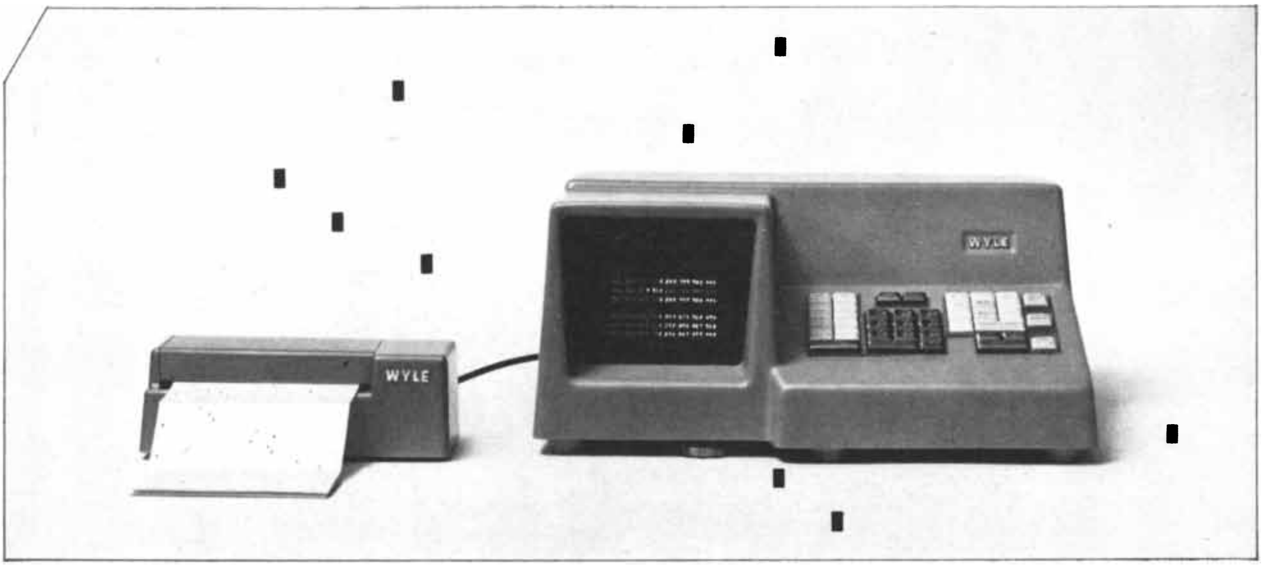

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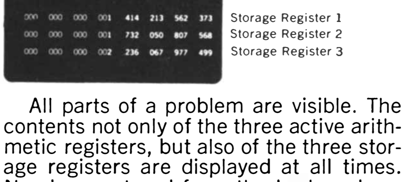

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LETTERS

Sirs:

The critical momentum of his review of Bernard Berelson and Gary A. Steiner's *Human Behavior: An Inventory of Scientific Findings* [Scientific American, July] carries Jules Henry into certain errors concerning the study of human and animal behavior. He speaks of "the Skinner box, an automatic baby-tender that makes it unnecessary for parents to touch their babies." The device does nothing of the sort. It was designed to improve the physical environment of the baby. If it has any effect on the social environment, it is in furthering warmer, happier and more productive contacts between parent and child. But Henry has got the wrong box. He should be talking about a quite different piece of equipment used in the study of operant behavior. He refers to operant conditioning but confuses the reader by moving at once to an isolated phenomenon as if it characterized the whole field. The particular fact he refers to is this: if an apparatus delivers small amounts of food to a hungry organism, the organism develops a superstition in the sense that it begins to behave as if the delivery of the food were contingent on its behavior. It acts "as if it were causing the food to appear."

Professor Henry argues that to extrapolate this principle to superstition in man breaks "the law of homologous extrapolation" because rats and pigeons are not homologous with man. One would suppose that a very considerable homology has been established in the biological and medical sciences, much of which is relevant to behavior. Whether a fact about a lower organism also holds for man is a question to be decided not by anthropological fiat but by research. Many characteristic features of operant behavior hold for a surprisingly wide range of species, including primates and among them man. For example, certain subtle and complex schedules of reinforcement are formally homologous in the environments of men and animals and prove to have similar effects. The schedules built into gambling devices have been extensively studied in operant laboratories, and effects on rats and pigeons are surprisingly close to those to be seen in Las Vegas.

The reinforcing contingencies that generate superstitious behavior over a wide range of species do not explain all superstitious phenomena. The particular response adopted by a pigeon is due to a temporal accident, and two pigeons are not likely to hit on the same response. The fact that different Zuñi Indians appear to behave in more or less the same way suggests that the behavior is transmitted, and transmission requires its own explanation. The contingencies that first led to dancing in a given way, singing particular songs or making prayer sticks of a given kind are not now available for study. But the contingencies that maintain the transmitted forms can be examined. Zuñi Indians do dance as if their dancing produced rain. It is relevant that rain coming at the end of a long drought is strongly reinforcing, and that the more conspicuous and stereotyped the behavior with which it coincides, the more effective the contingencies. It is also relevant that aversive conditions that tend to be self-limiting are particularly favorable to the development of superstition. Recovery from illness, for example, is likely to be reinforcing, and short-term illnesses breed many supposed "cures."

Henry can no doubt support his claim that "extrapolation from animals to humans leads to absurdity sometimes compounded by ignorance." Anthropologists have abundantly demonstrated that the same can be said for extrapolations from one human group to another. But extrapolations may also have happier consequences. Etymology may restrain anthropologists from enjoying them,

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since their subject is *anthropos,* but they should not play dog in the manger—if an extrapolation from a lower organism may be forgiven.

B. F. SKINNER

Harvard University
Cambridge, Mass.

Sirs:

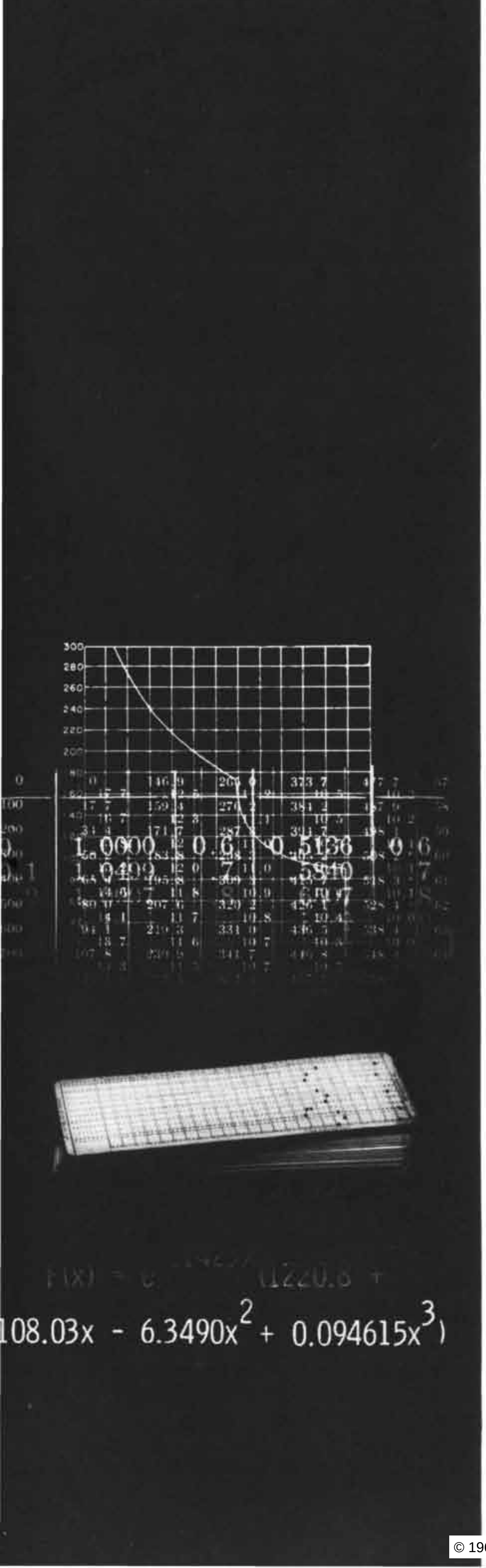
Nobody denies the importance of reinforcement. The owners of gambling casinos understood it long before psychologists made it the foundation of theory. This is not the issue in the extrapolation of operant-conditioning theory to man. In the pigeon case the positions assumed by the pigeon are determined by random (that is, largely unknown) neural impulses and reinforcement fixes a particular position. In man, however, *the idea comes first.* For example, bone-pointing in Australia is reinforced and sustained by the imagined effect of the pointing on the enemy —he is "killed." *But the idea of the pointing had to come first.* Pigeons have no ideas, hence it is a fatal error to say the pigeon "acts *as if* it were causing the food to appear." No—it merely looks that way to psychologists. The expression "as if" is inadmissible in scientific thinking when the units compared are as different as man and pigeon, but it is precisely "as if" thinking that has misled a generation of psychologists. It is a kind of magical thinking in its own right. The motto is "If the pigeon behaves as if it were a man, it is a man."

There is no magical process however simple that cannot be examined in the same context. My friends the Kaingang Indians, for example, shout at thunderstorms to make them go away. It is obvious that the continuance of the shouting is guaranteed by the fact that sudden squalls always go away—that is, shouting is "rewarded." But the *idea* of shouting was an invention—an idea.

My criticism of the book was based on the uncritical extrapolation of experimental results from animals to man and was not a criticism of learning theory in general. Learning theory has two simple points to make and does so with Talmudic ingenuity, variability, intricacy and insistence. They are reinforcement and extinction. What has to be left out, because the subjects are mostly animals, is thought.

JULES HENRY

Northwestern University
Evanston, Ill.



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The search for unknown relationships is basic to science and engineering . . . and results in a steady outpouring of new tables and graphs. To store this mass of data economically and retrieve it quickly from a computer, mathematicians suggest the use of formulas that closely approximate or "fit" the data.

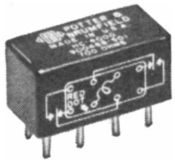
Here at the General Motors Research Laboratories, one of our four mathematical science departments has taken the first giant stride toward making such formulas easy to obtain. Through pioneering work in approximation theory, our mathematicians have been able to develop automatic computer procedures—"black boxes" that can crank out very efficient approximation formulas.

The formulas might be weighted polynomials . . . or the more flexible rational functions . . . or the little known, highly versatile spline functions. But in any case, their generation by delicately tuned computer programs goes well beyond standard "curve-fitting" techniques. In using these programs, for example, our scientists and engineers may ignore such mathematical subtleties as the Tchebycheff norm and unisolvence. Just feed the table in, pull a formula out.

A practical result of mathematical research, automatic approximations, we believe, well illustrate the exciting work going on in General Motors to make the computer a more efficient, more useful problem-solving tool.

General Motors Research Laboratories

Warren, Michigan



Who says a relay this small can't be reliable?

This relay was designed to be reliable. It's our HC Series . . . non-latching, non-polarized. Data compiled from a comprehensive testing program validate the design for use in critical applications.

Superior performance is obtained through the employment of bifurcated contacts . . . special materials not found in other similar relays . . . manufacturing tools and techniques in step with the state-of-the-art. A vigilant quality assurance program demands production within the scope of MIL-Q-9858A.

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HC ENGINEERING DATA

GENERAL: Non-polarized half crystal case size non-latching relay.

Shock: 50g for 11 ms.
Vibration: 20g to 3000 cps. } No contact opening in either armature position.

Operate Time: 3 milliseconds max. at nominal voltage @ +25°C coil temperature.

CONTACTS

Arrangements: DPDT (bifurcated, gold-plated silver-alloy).
Rated: Dry circuit to 2 amps at 28.0 VDC res.
Life: 100,000 operations at maximum rated load.
Temperature Range: –65°C to +125°C.
Size: .800″ long, .400″ wide, .400″ high (seated).
Weight: Approx. 1/4 oz.

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Today's RCA 3301 computer...with a full spectrum of communications and peripheral devices, plus unparalleled low-cost mass random access memory...outvalues every competitive system, including those announced for delivery years from now.

Consider all the things you would like a computer to do for your business. Count up the advantages of shifting smoothly from business data processing to operations-wide communications to management science . . . to on-line, real-time control. Now you have a picture of a total information system . . . capable of decision-making guidance that increases your competitive edge and decreases your costs . . . all attainable with one modular RCA 3301 computer system.

The RCA 3301, deliverable this year, has the lowest cost-performance index ever achieved in commercial data processing. It creates a new generation of computer values. Its wide range of communications devices, drawing on RCA experience in military command-control systems, permits a flexible two-way data-flow between the computer and all your operating points, local and remote. Its array of peripherals meets every total-system requirement, as this sampling shows:

The 3488 RANDOM ACCESS MEMORY . . . stores 5.4 billion characters at the unprecedented low cost of 5¢ per month per 10,000 characters . . . with retrieval speed 60% higher than similar devices.

High-speed MAGNETIC DRUM STORAGE . . . available in six capacities, from 0.3 to 2.6 million characters, with average access time of 8.3 milliseconds.

Five choices of MAGNETIC TAPE STORAGE . . . including industry-compatible units with data transfer rates up to 120,000 characters per second.

Three types of VIDEO DATA TERMINALS . . . utilizing RCA television technology to display computer output at remote locations, via standard communication lines.

COMMUNICATIONS MODE CONTROL . . . enabling simultaneous on-line input-output of data between the computer and up to 160 branch locations, via leased telephone or telegraph lines—or public dial network.

DATA EXCHANGE CONTROL . . . linking two or more RCA computers memory-to-memory for more work-power on the same program.

EDGE . . . RCA's powerful electronic data gathering equipment, which can report every major production step to a central computer from multiple input stations throughout the plant.

Total management information systems are not a deferred promise of the future. They are an RCA reality today. The RCA 3301 concept of "functional modularity" allows this single system to expand in function and capacity over the broadest growth radius, answering your computer needs for years ahead. With a stepped-up memory cycle time of 1.5 microseconds . . . the availability of every required peripheral . . . comprehensive software . . . shortened delivery schedules . . . and pricing to encourage total systems in more of today's business and government . . . the RCA 3301 is your best computer buy.

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The Most Trusted Name in Electronics

50 AND 100 YEARS AGO

Scientific American

SEPTEMBER, 1914: "To appreciate the stupendous character of the War of the Nations which is now in full swing on the continent of Europe, we must bear in mind two facts: first, that it is a war to the death; second, that in the full realization of the absolute finality of the result, every one of the contending nations has already called out, or has stated that it will do so, the whole of its trained reserves, thus putting some 16 millions of men under arms. The one mournful consolation to be drawn from this unspeakable calamity is to be found in the belief that the loss of life, the destruction of property, the paralysis of trade and industry and the total setback of civilization will be so stupendous as to bring the nations of the earth together, when the war is over, in the endeavor to substitute for the present brutal armaments an International Tribunal backed by an International Military Police."

"At the beginning of the war it was well understood (for a distinguished German general had written it all out in a book) that the grand strategy of Germany would be to concentrate the flower of her army against the French, overwhelm and scatter them in the first two or three weeks of the war, invest Paris and then turn against Russia in an equally swift and crushing onslaught. At the present writing, with the war over a month in progress, nothing of the kind has happened. The French and British armies are intact and everywhere in touch; they have fallen back upon the line of forts and strongly entrenched positions provided years ago by France to meet the inevitable German attack; for the past three weeks they have not only presented an impregnable wall of defense, through which the Germans have failed to break, but also in the rearward movement to their present position they have fought a series of defensive battles in selected and entrenched positions, which, on the theory that the losses are three to one against the attacking force, must have cost the German army already not less than 200,000 men in killed, wounded and otherwise disabled."

"Ernest Solvay, known the world over for his remarkable soda process and one of the wealthiest industrialists in Europe, is being held by the Germans as a hostage for the payment of the $40,000,000 war levy imposed on Brussels by the Germans."

"A Berlin paper states that Prof. Roentgen does not care to retain the English medal presented to him by the Royal Society in recognition of his discovery of X-rays and has therefore given it to the Red Cross. The medal contains $250 worth of gold."

"The South Magnetic Pole has not yet been definitely located, according to recent reports from Dr. Mawson's Australian Antarctic expedition. Heretofore it has been assumed that Dr. David visited this pole in the course of Shackleton's expedition in 1909, but according to Mawson's recent investigations the spot reached by David, although lying within the region in which the magnetic needle is at times vertical, is not the center of that region, which appears to be of much greater extent than was formerly supposed. Moreover, there appear to be several local poles around the principal pole."

"A discussion of a systematic search for nebulae in Kapteyn's 'selected areas' by E. A. Fath is given in the *Astronomical Journal,* plates having been taken covering 139 out of the 206 regions comprising the scheme. The exposures were made with the 60-inch reflector at Mount Wilson, each lasting one hour. A number of new nebulae are noted and an interesting series of conclusions as to the general distribution is given. The concentration of nebulae near the north galactic pole is very noticeable, but there are great variations in the density of distribution. There is a complete absence of these nebulae where the Milky Way crosses the sky."

Scientific American.

SEPTEMBER, 1864: "The recent naval engagement at Mobile Bay is a convincing proof of the views we have always expressed regarding the invulnerability and utility of the monitors.

From the published account it appears that Admiral Farragut recognizes their value; he sent one of them forward—the *Tecumseh*—and had others of the iron-clad fleet in his squadron. The rebel ram *Tennessee* was disabled only after a long fight, during which the wooden vessels were of no value whatever. The shots that penetrated the rebel iron-clad were all from the 15-inch guns on the monitors, while the broadsides so bravely poured in by the flag-ship *Hartford* rebounded from the iron armor without injuring it in the least. None can admire more than ourselves the skill, courage and coolness of Admiral Farragut and we cheerfully add our voice to the praise showered upon him from all sides; but let us hear no more of the 'worthless monitors.' Iron hearts in wooden ships are undoubtedly good, but iron hearts in iron ships are better still."

"A correspondent of the Cincinnati *Commercial,* writing from near Atlanta, says:—'Among the many modifications and new features of warfare that have been introduced during the present struggle, the one most noticeable and most revolutionizing in its tendency is the practice becoming so universal on both sides of *intrenching and fortifying.* "Open-field fighting" has almost passed into history. It only occurs now when the party attacked is either surprised or flanked. Experience has abundantly shown that no general can afford very often to storm well-manned breastworks, for although he may carry his point, yet he does it at an enormous sacrifice of life and a sacrifice generally greatly disproportioned to the injury he inflicts.' "

"M. Lemaire denies that a special ferment for every kind of fermentation exists. He finds the same microscopic beings present whether sugar is being changed into alcohol or alcohol into acetic acid. But in the case of natural animal and vegetable matters he has assured himself that microzoa begin the decomposition, which, when the matters become acrid, is carried on by microphytes. By means of a little acid these latter can be made to appear at will and the author consequently argues that mycoderms do not make the acid but appear in consequence of its presence."

"Mr. Nobel announces that by damping mining powder with nitro-glycerine its explosive power is trebled and the noise of the explosion is much less than when ordinary powder is used."

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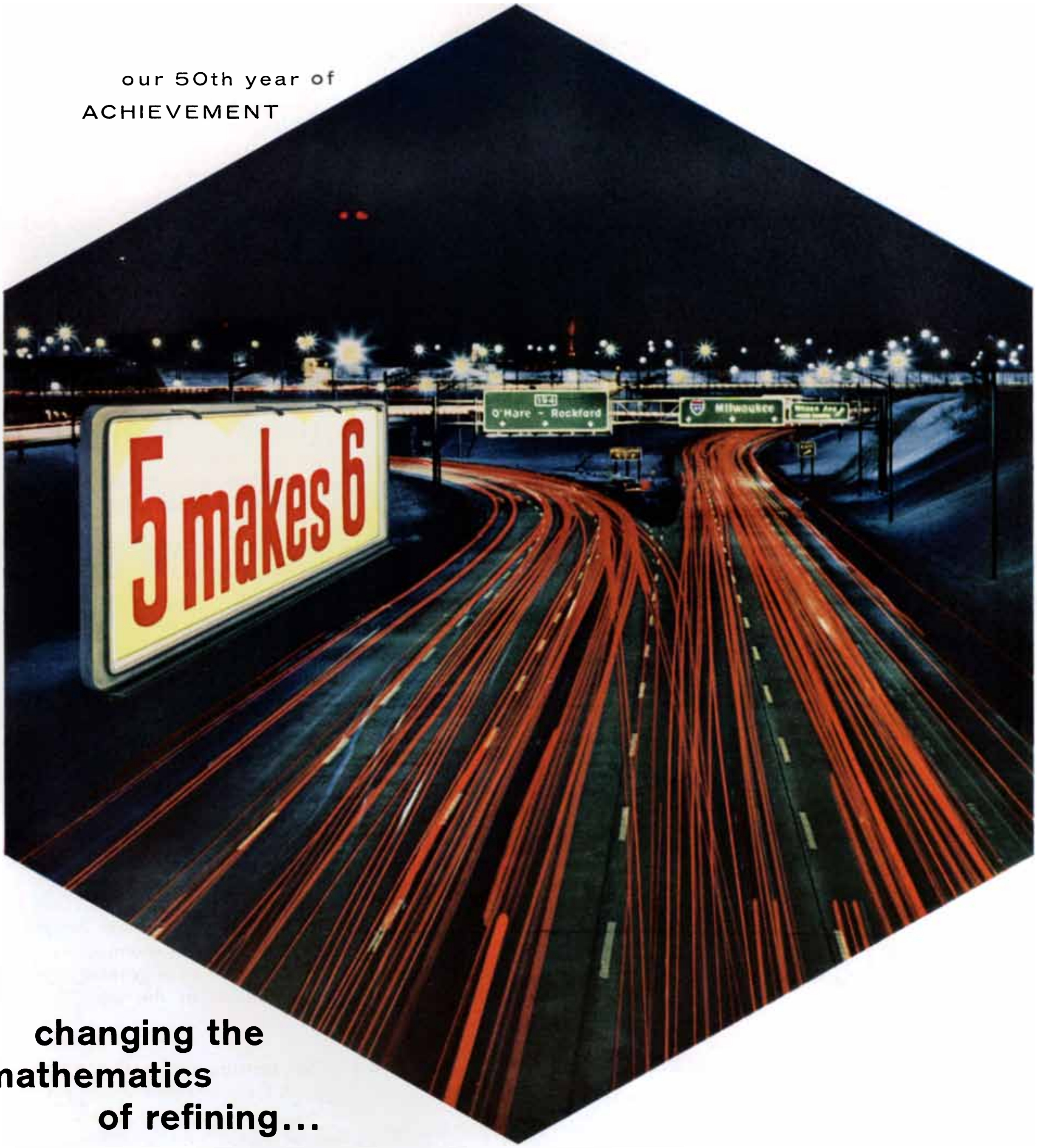

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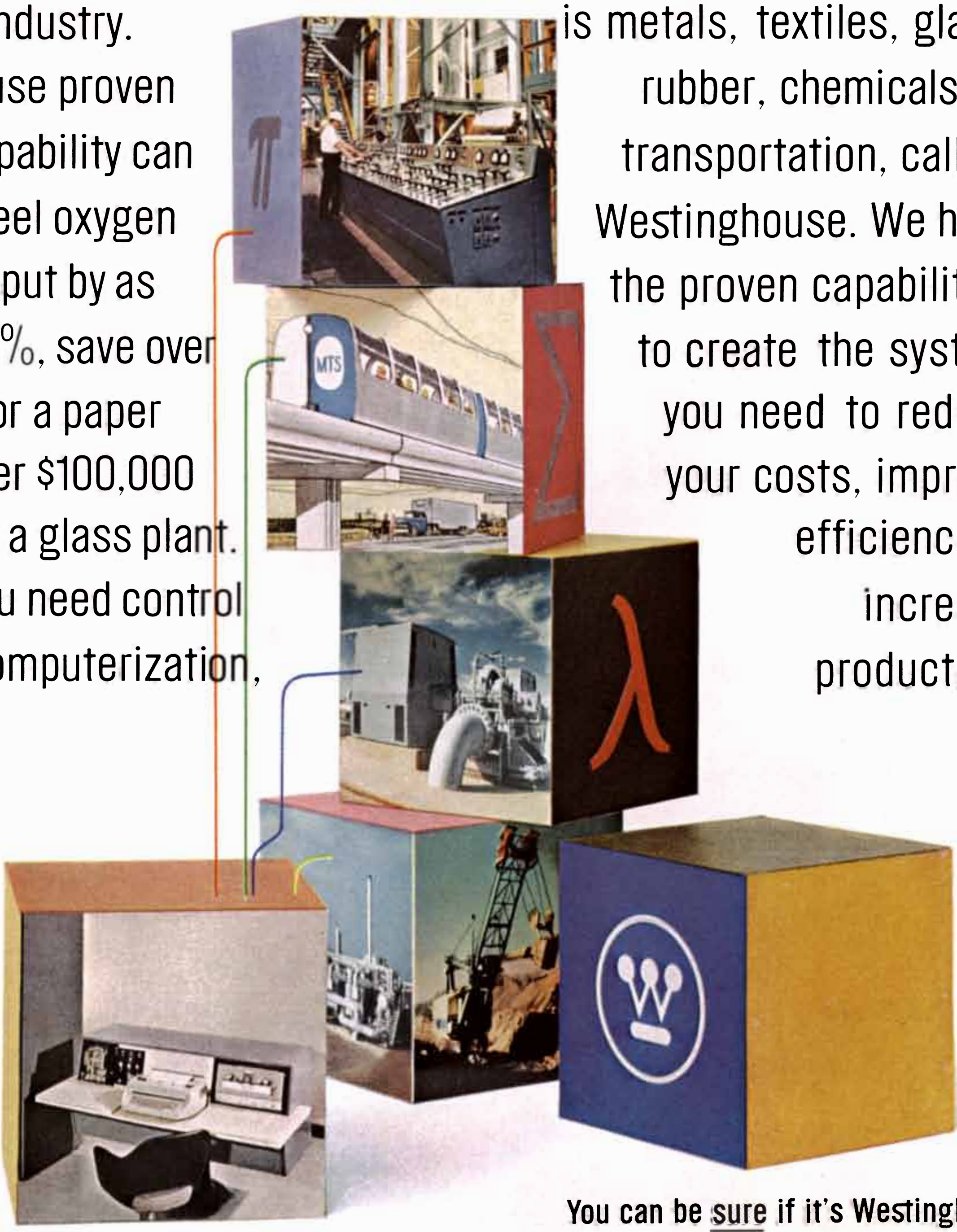

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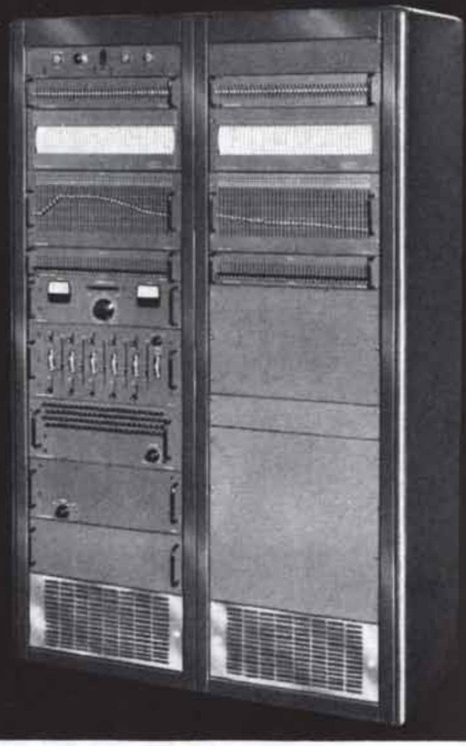

L T V L I N G E L E C T R O N I C S D I V I S I O N

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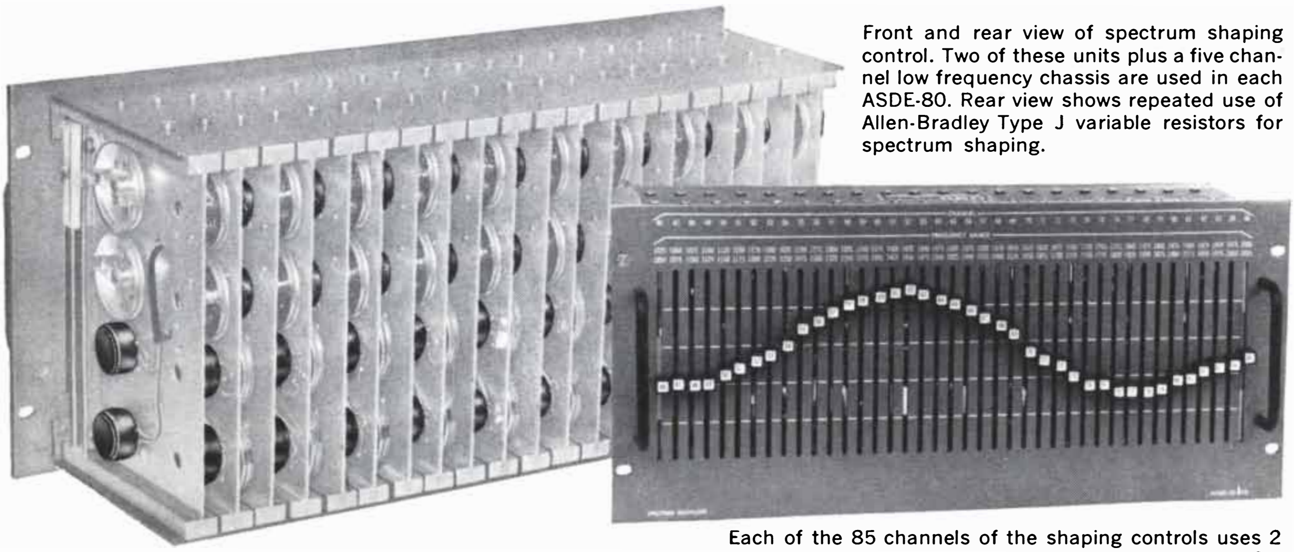
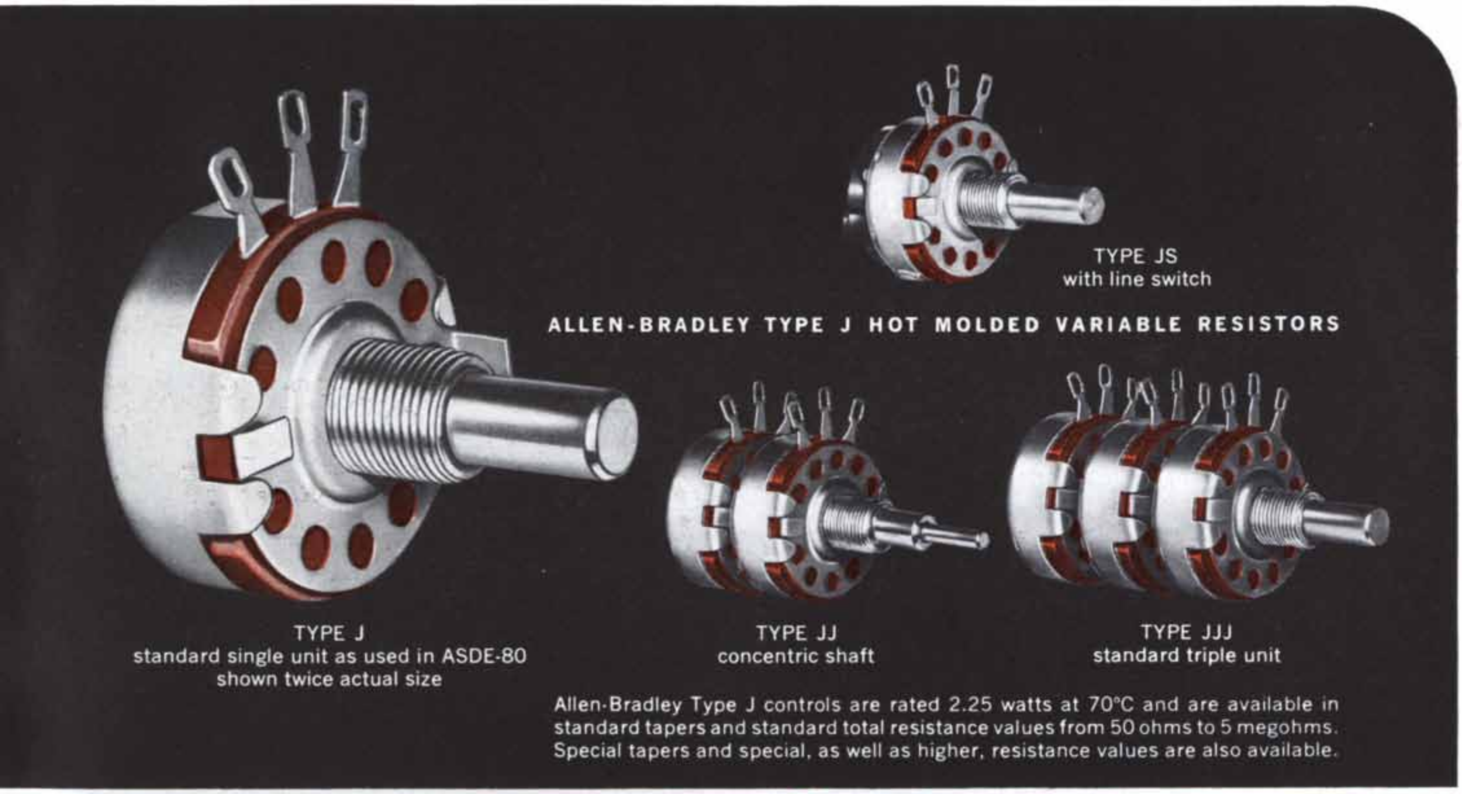


Front and rear view of spectrum shaping control. Two of these units plus a five channel low frequency chassis are used in each ASDE-80. Rear view shows repeated use of Allen-Bradley Type J variable resistors for spectrum shaping.

Each of the 85 channels of the shaping controls uses 2 Type J variable resistors—a total of 170 in each ASDE-80.



Since switching to exclusive use of Allen-Bradley Type J *hot molded* variable resistors—shortly after introducing their first equalizers in 1956—LTV Ling Electronics Division has achieved improved performance and reliability.

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THE AUTHORS

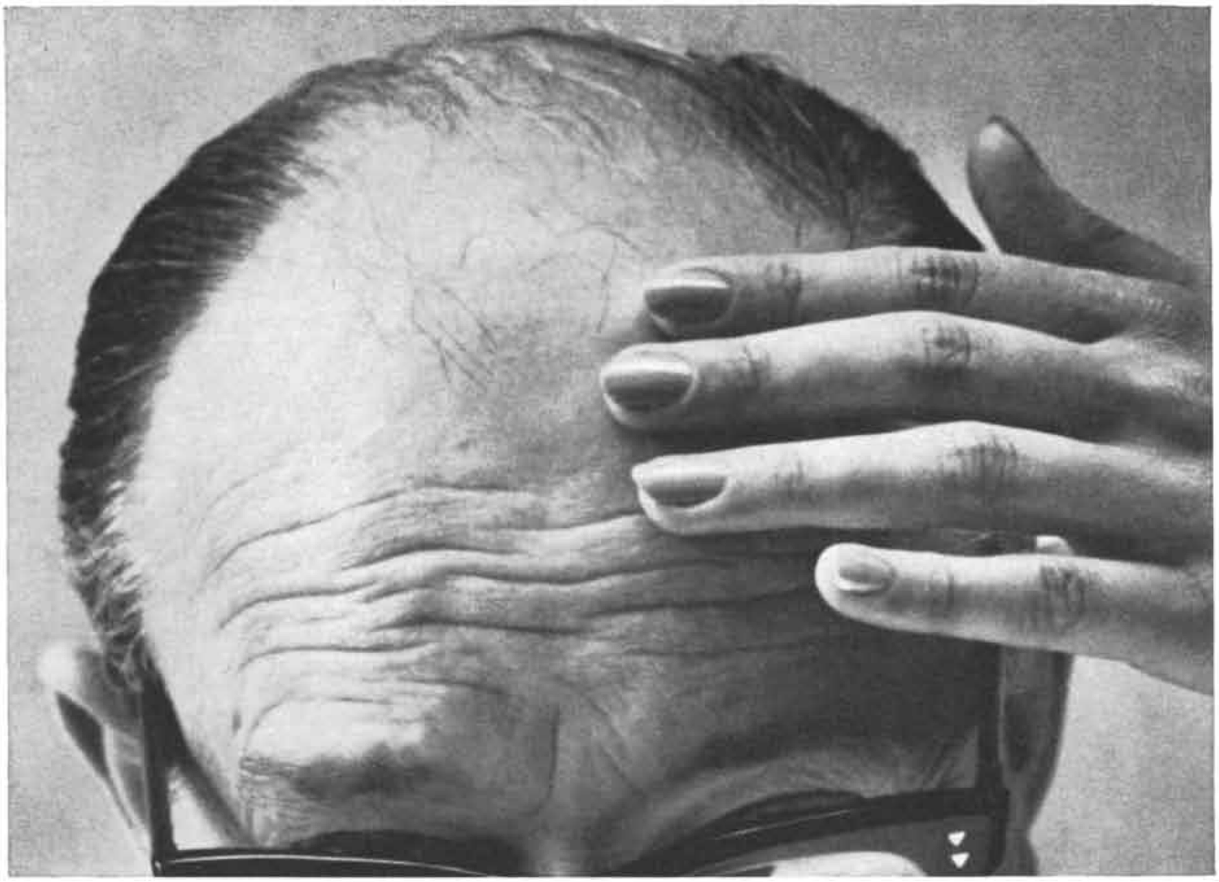
RICHARD COURANT ("Mathematics in the Modern World") is professor emeritus of mathematics at New York University. Courant was born in Lublinitz, Poland, in 1888 and was educated at the universities of Breslau, Zurich and Göttingen, receiving a Ph.D. from the last-named institution in 1910. He taught at Göttingen and at the University of Münster until 1920, when he was appointed professor of mathematics and director of the Mathematics Institute at Göttingen. In 1932 he left Göttingen to serve short terms as visiting professor of mathematics at the University of California, Princeton University and the University of Cambridge. He joined the N.Y.U. faculty in 1934 and in 1936 became head of the mathematics department and director of the Institute of Mathematical Sciences at N.Y.U. He served in these capacities for 22 years until his retirement in 1958. In 1961 the Institute of Mathematical Sciences was renamed the Courant Institute in his honor. Courant has written six books, including *Methods of Mathematical Physics* (with David Hilbert) and *What Is Mathematics?* (with Herbert Robbins).

PHILIP J. DAVIS ("Number") is professor of applied mathematics at Brown University. He obtained a B.S. and a Ph.D. in mathematics from Harvard University in 1943 and 1950 respectively. From 1952 to 1963 he worked in the Applied Mathematics Division of the National Bureau of Standards. Davis was awarded the Chauvenet Prize of the Mathematical Association of America in 1963 for his paper "Historical Profile of the Gamma Function." He is the author of two books: *The Lore of Large Numbers*, published in 1961, and *Interpolation and Approximation,* published in 1963. At present he is working in the field of numerical analysis, which is concerned with the development of techniques for the solution of equations on high-speed computers.

MORRIS KLINE ("Geometry") is professor of mathematics and chairman of the undergraduate mathematics department at the Washington Square College of New York University. He is also director of the Division of Electromagnetic Research at N.Y.U.'s Courant Institute of Mathematical Sciences. After receiving a Ph.D. from N.Y.U. in 1936 Kline spent two years doing research at the Institute for Advanced Study in Princeton, N.J. He taught mathematics at N.Y.U. from 1938 to 1942, when he joined the U.S. Army as a physicist at the Signal Corps Engineering Laboratories. He returned to N.Y.U. in 1945 and has been a member of the faculty there ever since. During the academic year 1958–1959 Kline was a Fulbright lecturer at the Technische Hochschule in Aachen, Germany. In addition to his numerous technical publications Kline has written several popular books on mathematics, including *Mathematics in Western Culture,* published in 1953, *Mathematics and the Physical World,* published in 1959, and *Mathematics: A Cultural Approach,* published in 1962. This fall he will teach a course entitled "Mathematics in Western Culture," which will appear on the CBS-TV Sunrise Semester program. The special feature of the course, he writes, "is that it is an attempt to show that one can give a college freshman course in mathematics that stresses the cultural relations and implications of mathematics. I am hoping that this course will make some contribution toward relating the two cultures, to use C. P. Snow's phrase."

W. W. SAWYER ("Algebra") is professor of mathematics at Wesleyan University. Sawyer was born in England in 1911 and was graduated from the University of Cambridge, where he specialized in the mathematics of quantum theory and relativity. He taught mathematics at University College in Dundee, Scotland, and at the University of Manchester from 1935 to 1945, when he was appointed head of the mathematics department at the Leicester College of Technology. In 1948 he became head of the mathematics department at the University of Ghana (then University College, Gold Coast). From 1951 to 1956 he was a member of the faculty of the University of Canterbury in Christchurch, New Zealand. He taught for a year at the University of Illinois before joining the Wesleyan faculty in 1958. Sawyer is the author of several popular books on mathematics, including *Mathematics in Theory and Practice, Prelude to Mathematics* and *Mathematician's Delight.*

MARK KAC ("Probability") is professor of mathematics at the Rockefeller Institute. Kac was born in Krzemieniec, Poland, in 1914 and educated at John Casimir University in Lwow, where he



received a Ph.D. in 1937. He came to the U.S. in 1938 as a Parnas Foundation Fellow at Johns Hopkins University and a year later joined the faculty at Cornell University. During the academic year 1951–1952 he was a member of the Institute for Advanced Study in Princeton, N.J. He left Cornell in 1961 to take up his present post. Kac was awarded the Chauvenet Prize of the Mathematics Association of America in 1950 for his paper "Random Walk and the Theory of Brownian Motion." His main scientific activity for the past few years has been in the field of statistical mechanics; he is particularly interested in the study of mathematical models of changes of phase.

W. V. QUINE ("The Foundations of Mathematics") is Edgar Pierce Professor of Philosophy at Harvard University. He was graduated in 1930 from Oberlin College, where he majored in mathematics. Two years later he obtained a Ph.D. from Harvard, having written his dissertation in logic under Alfred North Whitehead. After a year of informal study at the universities of Vienna, Prague and Warsaw, Quine was elected to Harvard's Society of Fellows in 1933. He began teaching at Harvard in 1936 and in 1948 became professor of philosophy and Senior Fellow in the Society of Fellows. Since then he has also lectured and studied at the University of Oxford, the Institute for Advanced Study in Princeton, N.J., and the Center for Advanced Study in the Behavioral Sciences in Palo Alto, Calif. In the summer of 1959 he lectured in Japan and Australia. Of the eight books he has written, the latest, *Set Theory and Its Logic,* was published in 1963. Quine was the author of the article entitled "Paradox" in the April 1962 issue of SCIENTIFIC AMERICAN.

FREEMAN J. DYSON ("Mathematics in the Physical Sciences") is professor in the School of Mathematics of the Institute for Advanced Study in Princeton, N.J. A native of England, Dyson was educated at Winchester College and the University of Cambridge. His undergraduate career at Cambridge was interrupted by a wartime stint in the Royal Air Force, where his mathematical talents were employed in investigating the cause of the heavy loss of bombers on night missions. After reading the Smyth report on atomic energy in 1945 he decided to return to Cambridge to study physics. He was a fellow at Trinity College, Cambridge, for a year and then came to the U.S. on a Commonwealth Fund Fellowship to study at Cornell University with Hans A. Bethe and Richard P. Feynman. After two years as professor of physics at Cornell he joined the Institute for Advanced Study in 1953. For the past 10 years he has worked on various problems lying on the border line between physics and mathematics, particularly in quantum field theory and in the statistical mechanics of complex systems. Of the three other articles Dyson has written for SCIENTIFIC AMERICAN, the latest, "Innovation in Physics," was published in September, 1960.

EDWARD F. MOORE ("Mathematics in the Biological Sciences") is a mathematician in the switching research department of the Bell Telephone Laboratories in Murray Hill, N.J. A graduate of the Virginia Polytechnic Institute, Moore acquired a Ph.D. in mathematics from Brown University in 1950. After working on digital-computer programming at the National Bureau of Standards and the University of Illinois, he joined the staff of the Bell Telephone Laboratories in 1951. During the academic year 1961–1962 he was simultaneously Gordon McKay Visiting Lecturer on Applied Mathematics at Harvard University and visiting professor of electrical engineering at the Massachusetts Institute of Technology. Moore has done research on a wide variety of computers, ranging from game-playing machines to machines for designing switching circuits. His chief current interest is in the theoretical capabilities and limitations of automata. This is Moore's second article for SCIENTIFIC AMERICAN; the first, "Artificial Living Plants," appeared in October, 1956.

RICHARD STONE ("Mathematics in the Social Sciences") is P. D. Leake Professor of Finance and Accounting at the University of Cambridge. Born in London in 1913, Stone studied law and economics at Gonville and Caius College, Cambridge. During World War II he did statistical work for the War Cabinet and also served as assistant to John Maynard Keynes at the Treasury. In 1945 he was appointed the first director of the Department of Applied Economics at Cambridge. He acquired an Sc.D. from Cambridge in 1955. Stone's past work has been mainly in the fields of social accounting and econometrics. At present he is engaged, with a group of colleagues at Cambridge, in constructing a computer model of the British

economy designed to study the possibilities of stimulating economic growth. He is also working on an international survey of economic planning models.

RICHARD BELLMAN ("Control Theory") is a mathematician on the staff of the RAND Corporation. A graduate of Brooklyn College, Bellman received a Ph.D. in mathematics from Princeton University in 1946. During World War II he taught radar in the Air Force, worked on sonar for the Navy and served the Army as a member of the Special Engineering Division working at Los Alamos on the atomic bomb. After the war he taught for two years at Princeton before joining the faculty of Stanford University in 1948. He left Stanford in 1951 to work on thermonuclear weapons at Princeton as a member of Project Matterhorn. He joined the RAND Corporation in 1952. Bellman is the author of 17 books and more than 375 published research papers on a wide range of mathematical topics.

STANISLAW ULAM ("Computers") is a research adviser at the Los Alamos Scientific Laboratory of the University of California. A native of Lwow, Poland, Ulam received an M.A. and a D.Sc. in mathematics from the Polytechnic Institute at Lwow in 1932 and 1933 respectively. He lectured at various institutions in Poland, England and France before coming to the U.S. in 1936 as a visiting member of the Institute for Advanced Study in Princeton, N.J. Shortly thereafter he became a fellow of the Harvard University Society of Fellows. He left Harvard in 1940 to join the faculty of the University of Wisconsin. Since going to Los Alamos in 1943 to work on the atomic bomb as a member of the Manhattan Engineer District, Ulam has taught for short terms at the University of Southern California, Harvard, the Massachusetts Institute of Technology, the University of Colorado and the University of California at San Diego. At Los Alamos, Ulam collaborated with Edward Teller on the development of the hydrogen bomb. He also invented the so-called Monte Carlo method, a procedure for finding solutions to mathematical and physical problems by random sampling. This technique, made practical by the development of high-speed computers, permits the solution of problems not amenable to more orthodox methods of analysis. Ulam is the author of *A Collection of Mathematical Problems*, published in 1960.

The man who said "you can't take it with you" was born a long time before Garrett started making life support systems.

As a matter of fact, unless man *does* take his earthly environment with him into space, he hasn't got a chance.

For here is a world that has no oxygen, no pressure, no gravity.

To live and work for weeks and months in orbital flight – a need dictated by urgent space projects now in progress – man must have the most sophisticated life support system ever built.

It has to provide him with oxygen, water, pressurization – complete climate control.

It has to guard him against temperatures that range from near absolute zero to the re-entry heat of thousands of degrees.

It has to be a miracle package.

The question becomes: Who is now building such an environmental system?

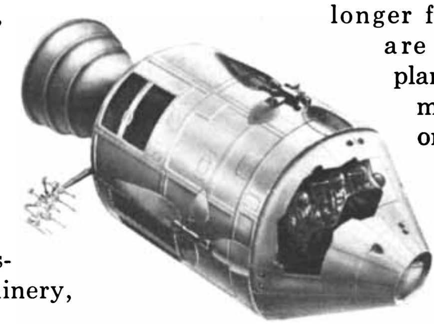
The answer is, of course, Garrett.

As long ago as 1941, Garrett management saw that man could go no higher, no faster, without pressurization of aircraft. When Garrett's AiResearch division delivered the first systems, suddenly the ceiling was off the world.

Today over 90% of the free world's aircraft carry Garrett environmental systems. Millions of hours of operation have been accumulated by heat transfer equipment, turbomachinery, controls.

This experience led Garrett to build the life support systems that protected our astronauts on the recent Mercury flights.

The same know-how is now at work supplying "shirtsleeve" environments for Gemini and Apollo. These systems will keep man alive for weeks in space. Now longer flights are being planned – manned orbiting laboratories and space stations. Garrett already knows how to solve life support problems for months in space. Much of the system work is completed and components built.



What are the reasons for this unique capability?

The most experienced men are Garrett men. The most advanced facilities are Garrett. The only applied system for outer space is Garrett built.

When the problem is environmental, the solution comes from Garrett because . . .

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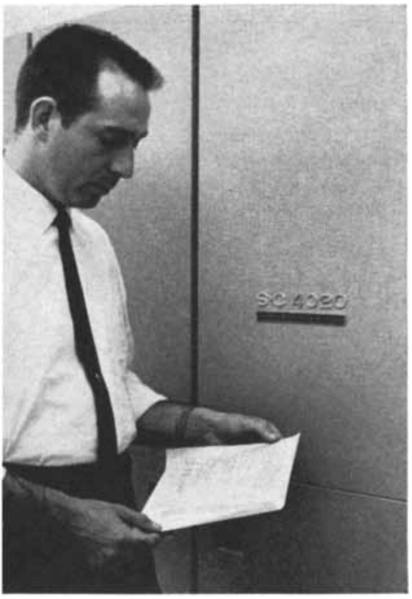
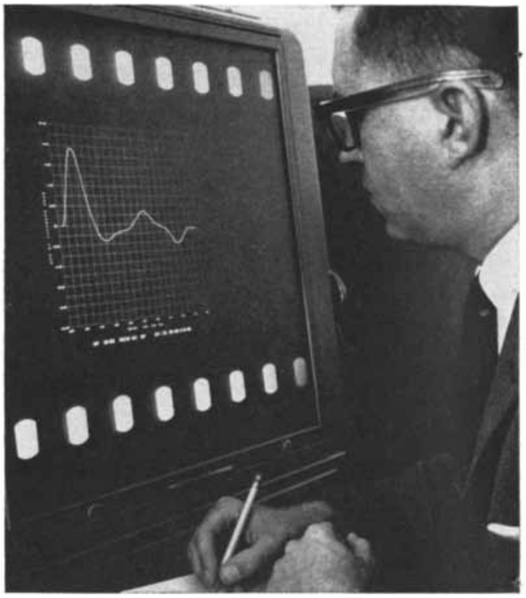


fast glance,

long hard look,



permanent film record

Already proven to be the most versatile computer output device available in terms of what it will produce—the S-C 4020 Printer/Plotter is equally as versatile in terms of how it will produce these various outputs.

For example, the *quick look* capability allows an operator to check any selected page of a program in seconds without the normal turn-around time associated with developing microfilm or photo-recording paper. This option employs a compact automatic developer together with a mechanical paper-cutting and stacking unit.

After the operator has produced selected *quick look* paper copies for program evaluation, the entire program may be run through a regular developing cycle where high-quality pages are printed for use in report distribution or paper records. In addition, the same computer output can be recorded automatically on 16 mm or 35 mm microfilm by the S-C 4020 for permanent storage and rapid retrieval.

GRAPHIC OUTPUT

The S-C 4020 not only records alphanumeric printing, it also converts and records digital output in combinations of curves, straight lines and characters. To convert digital codes into more easily interpreted graphics, tapes from a large-scale computer (typically a 7090 or 7094) are fed through the S-C 4020. The numerical language from the computer is translated and displayed on a CHARACTRON® Shaped Beam Tube, where it is transferred optically to page-size photosensitive paper and/or microfilm.

SPEED

The S-C 4020 will accept data from magnetic tape at input rates up to 62,500 six-bit characters per second. It will record this data on film at speeds in excess of 17,000 alphanumeric or symbolic characters per second. Frames combining characters, vectors and curves vary with the complexity of the specific drawing, but an average annotated graph can be recorded by the S-C 4020 in less than a second.

ECONOMICS

The S-C 4020 can be leased or purchased. Also, service bureaus are in operation on east and west coasts where you can lease 4020 time and try some of your own problems on the machine for evaluation. The S-C 4020 is paying its way many times over in more than a score of computer labs throughout the world. The equipment's high density input tape adapter results in minimum use of computer time for tape preparation.

APPLICATIONS

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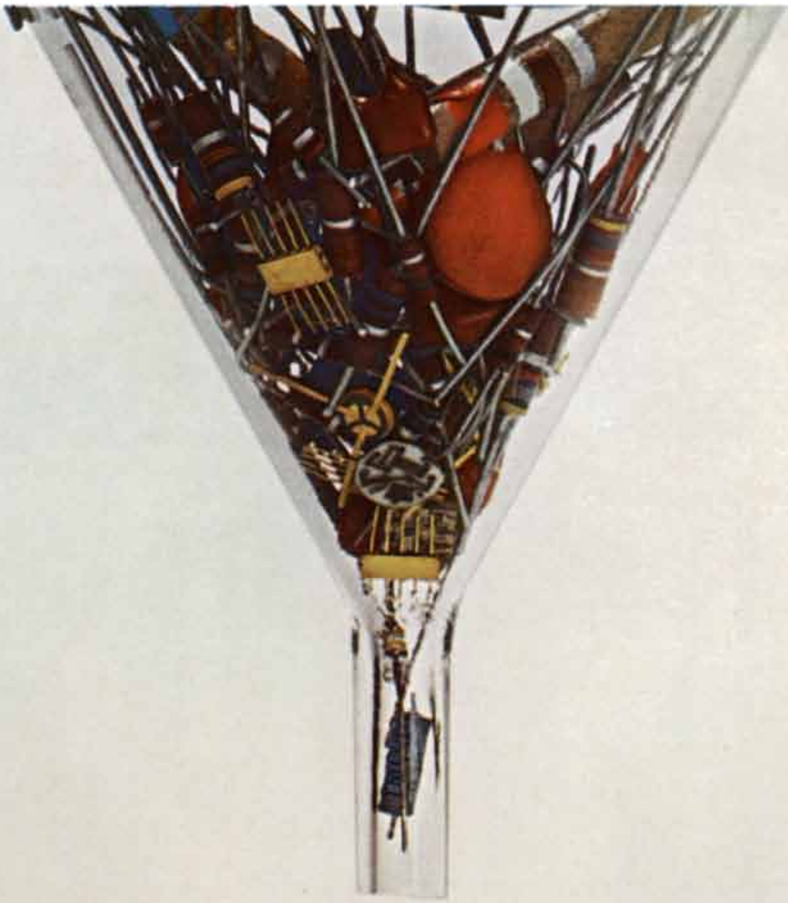

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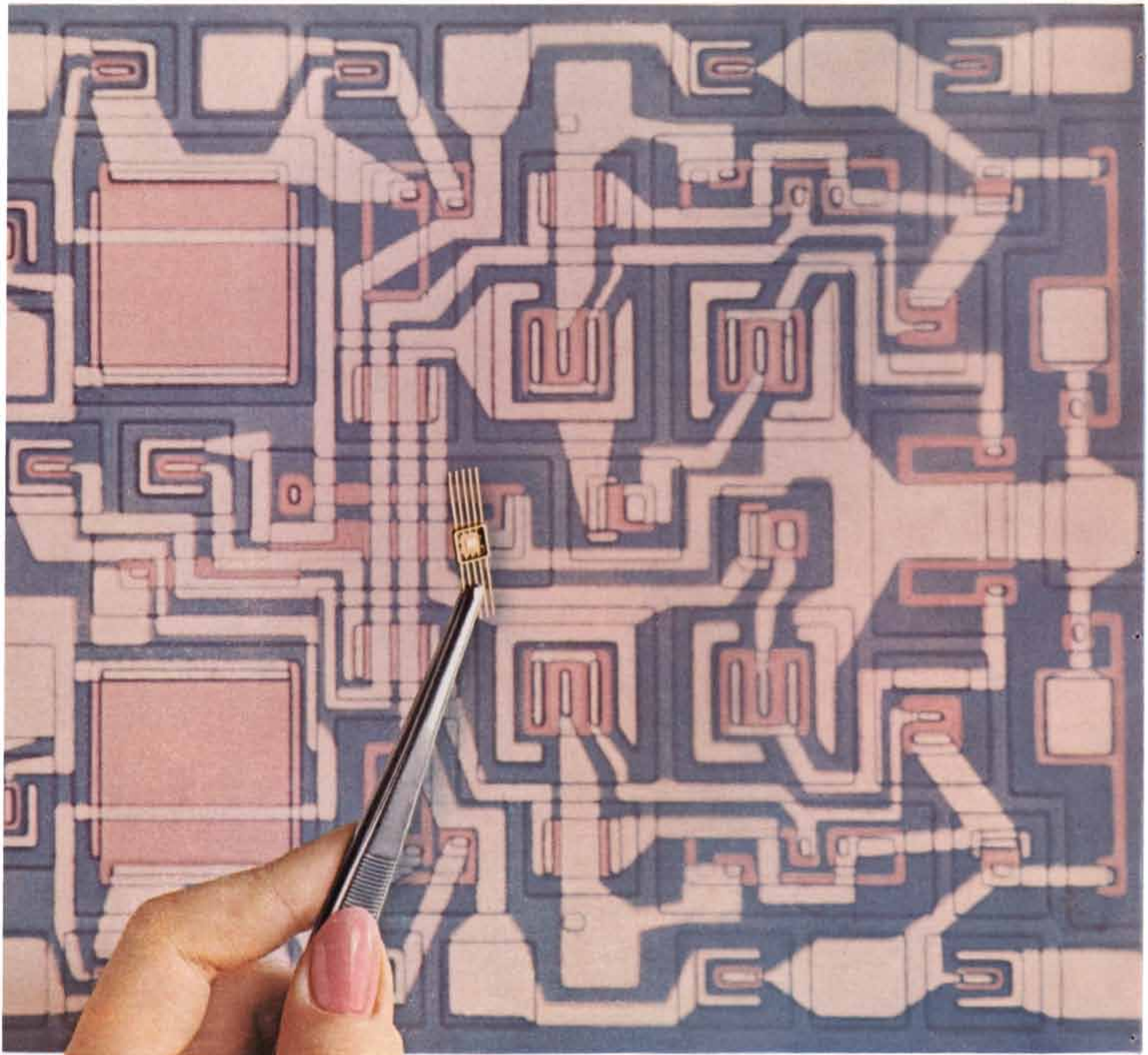

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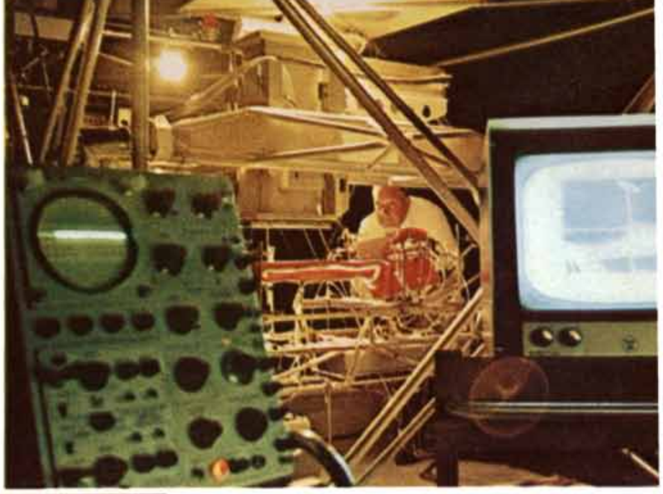

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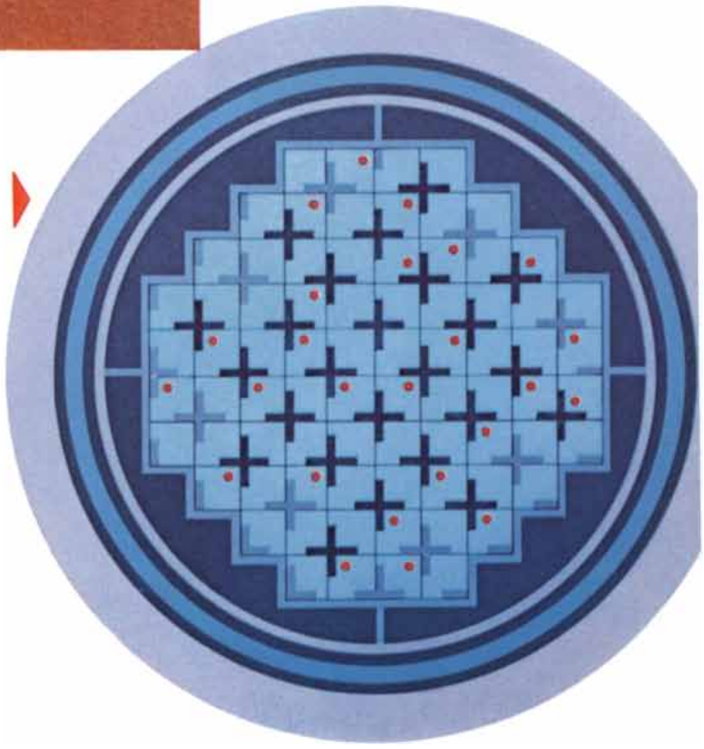

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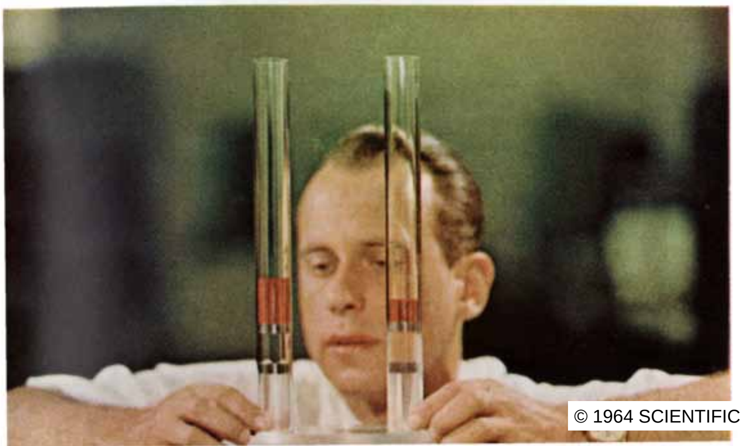

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Mathematics in the Modern World

Presenting an issue on mathematics in both its pure and applied aspects. Although the two are sometimes separated, a line cannot be drawn between them, and they traditionally fructify each other

by Richard Courant

The expanding role of mathematics in the modern world is vividly reflected in the proliferation of mathematicians. Since 1900 memberships in the several professional mathematical organizations in the U.S. have multiplied by an estimated 30 times. Today the number qualified by the doctorate stands at 4,800. During the past 25 years the number of mathematicians at work outside the universities in industry and Government has increased twelvefold. Activities of a more or less mathematical character now employ tens of thousands of workers at all levels of competence. In the colleges three times as many undergraduates were majoring in mathematics in 1962 as in 1956. Mathematics is no longer the preoccupation of an academic elite; it is a broad profession attracting talented men and women in increasing numbers. The scope of mathematical research and teaching has been greatly extended in the present period, and mathematical techniques have penetrated deep into fields outside the mathematical sciences such as physics, into new realms of technology, into the biological sciences and even into economics and the other social sciences. Computing machines and computing techniques have stimulated areas of research with obviously enormous and as yet only partly understood importance for mathematics itself and for all the sciences with inherent mathematical elements.

The contemporary role of mathematics is best appraised, however, by comparison with previous stages in its development. As recently as three centuries ago the main fabric of mathematical thought was supplied by geometry, inherited from the ancients and only meagerly augmented during the intervening 20 centuries. Then began a radical and rapid transformation of mathematics. The rigorous, axiomatic, deductive style of geometry yielded to inductive, intuitive insights, and purely geometric notions gave way to concepts of number and algebraic operations embodied in analytic geometry, the calculus and mechanics. It was the small intellectual aristocracy of the new mathematics that now spearheaded the forward thrust of science. By the time of the French Revolution the accumulated wealth of results and the demonstrated power of the mathematical sciences brought a widening of the narrow human basis of scientific activity, with the writing of textbooks to make the new mathematics more widely accessible, the systematic training of scientists and mathematicians in the universities and the opening up of new careers in the expansion of human knowledge.

The "classical" mathematics that had its beginnings in the 17th century retains its power and central position today. Some of the most fruitful work has come from the clarification and generalization of the two basic concepts of the calculus: that of function, which is concerned with the interdependence of two or more variables, and that of limit, which brings the intuitive notion of continuity within rigorous scrutiny. The concepts of mathematical analysis, including the theory of differential equations for one or more variables, which is an essential tool for dealing with rates of change, pervade the vastly extended territory of modern mathematics. That territory is surveyed in the next three articles in this issue of *Scientific American* from three points of vantage—number, geometry and algebra—

SECTION OF ANCIENT EGYPTIAN PAPYRUS on the opposite page reflects one of the earliest applications of mathematics: the measurement of land. Called the Rhind Papyrus after its modern discoverer, A. Henry Rhind, the entire work is a handbook of practical problems compiled about 1550 B.C. by a scribe named A'h-mosě. The horizontal lines separate five of the problems, which read from right to left. At the top is a section of "an example of reckoning area" of "a rectangle of land khet 10 by khet 2." Second from the top is a calculation of the area of a "round field" with a perimeter of khet 9. Other examples shown in the section calculate the area of triangular and trapezoidal fields. The title page of the papyrus describes it as a guide to "accurate reckoning of entering into things, knowledge of existing things all." The main portion of the papyrus is now in the British Museum.

that offer perspectives familiar to the nonmathematician. As will be seen, geometry has had a most fruitful growth, liberated by the concepts of function and of the number continuum; its youngest offspring, topology and differential geometry, rank among the most active and "modern" branches of mathematics. The special field of probability deserves a chapter in itself because it has found such wide application in science and technology and because it gives mathematical expression to some of the deep unsolved problems in the philosophy of science.

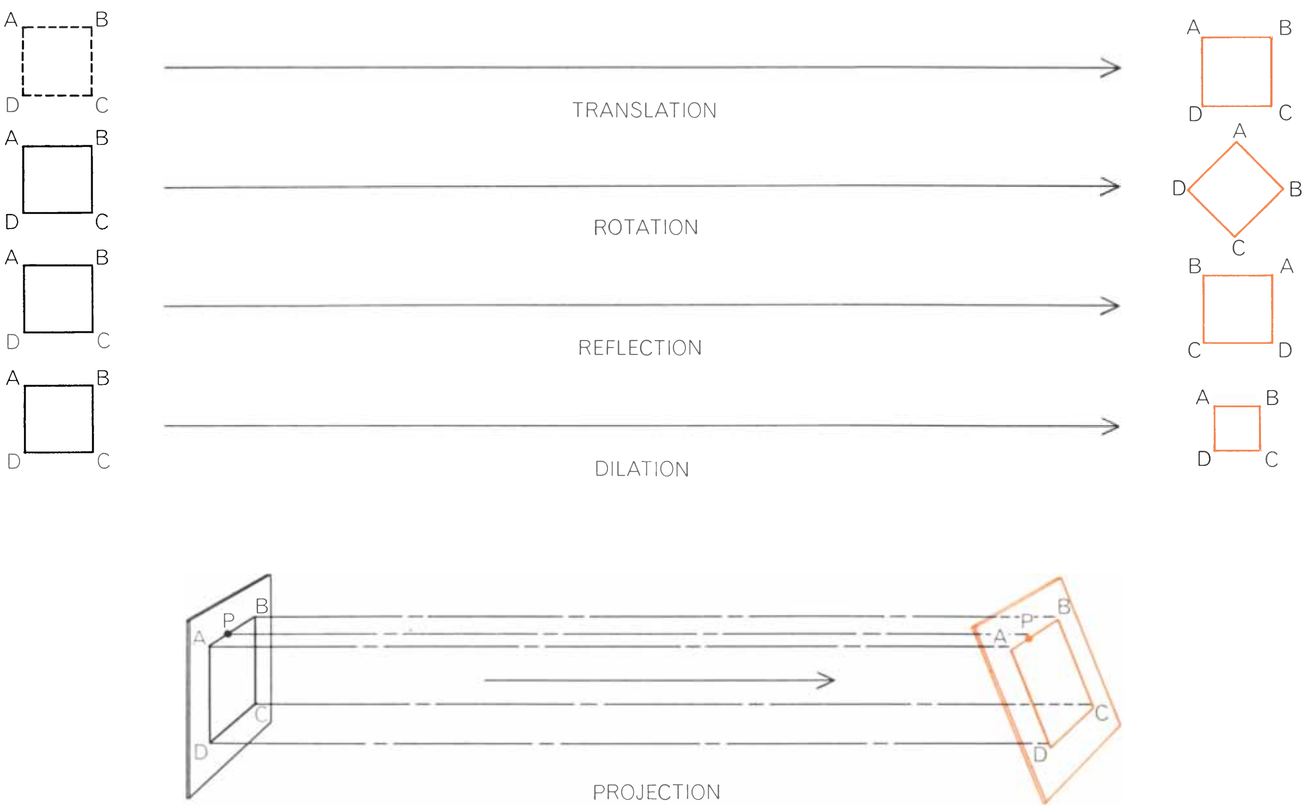
Mathematics today also reflects a vigorous trend, started early in the 19th century, toward solidification of the new conquests in the spirit of the mathematical rigor practiced by the ancients. This effort has inspired intense work on the foundations of mathematics, directed at clarifying the structure of mathematics and the meaning of "existence" for the objects of mathematical thought.

Inevitably the expansion of mathematics has enhanced immanent tendencies toward specialization and isolation; mathematics is threatened with a loss of unity and cohesion. Mutual understanding among representatives of different fields of mathematics has become difficult, and contact of mathematics with other sciences has been weakened. Yet remarkable advances continue to be won, mostly by young talent amply supported by a society that recognizes the increasing importance of mathematics. At the same time the growing volume of mathematical activity has led to a bewildering avalanche of publications, a multiplicity of meetings, administrative tangles and pressures of commercialism. It becomes the urgent duty of mathematicians, therefore, to meditate about the essence of mathematics, its motivations and goals and the ideas that must bind divergent interests together. For this purpose they can find no better occasion than the opportunity to explain their work to a wider public.

The question "What is mathematics?" cannot be answered meaningfully by philosophical generalities, semantic definitions or journalistic circumlocutions. Such characterizations also fail to do justice to music or painting. No one can form an appreciation of these arts without some experience with rhythm, harmony and structure, or with form, color and composition. For the appreciation of mathematics actual contact with its substance is even more necessary.

With this caution, some remarks of a general nature can nevertheless be made. As is so often said, mathematics aims at progressive abstraction, logically rigorous axiomatic deduction and ever wider generalization. Such a characterization states the truth but not the whole truth; it is one-sided, almost a caricature of the live reality. Mathematics, in the first place, has no monopoly on abstraction. The concepts of mass, velocity, force, voltage and current are all abstract idealizations of physical reality. Mathematical concepts such as point, space, number and function are only somewhat more strikingly abstract.

The model of rigorous axiomatic deduction for so long impressed on mathematics by Euclid's *Elements* constitutes the remarkably attractive form in which



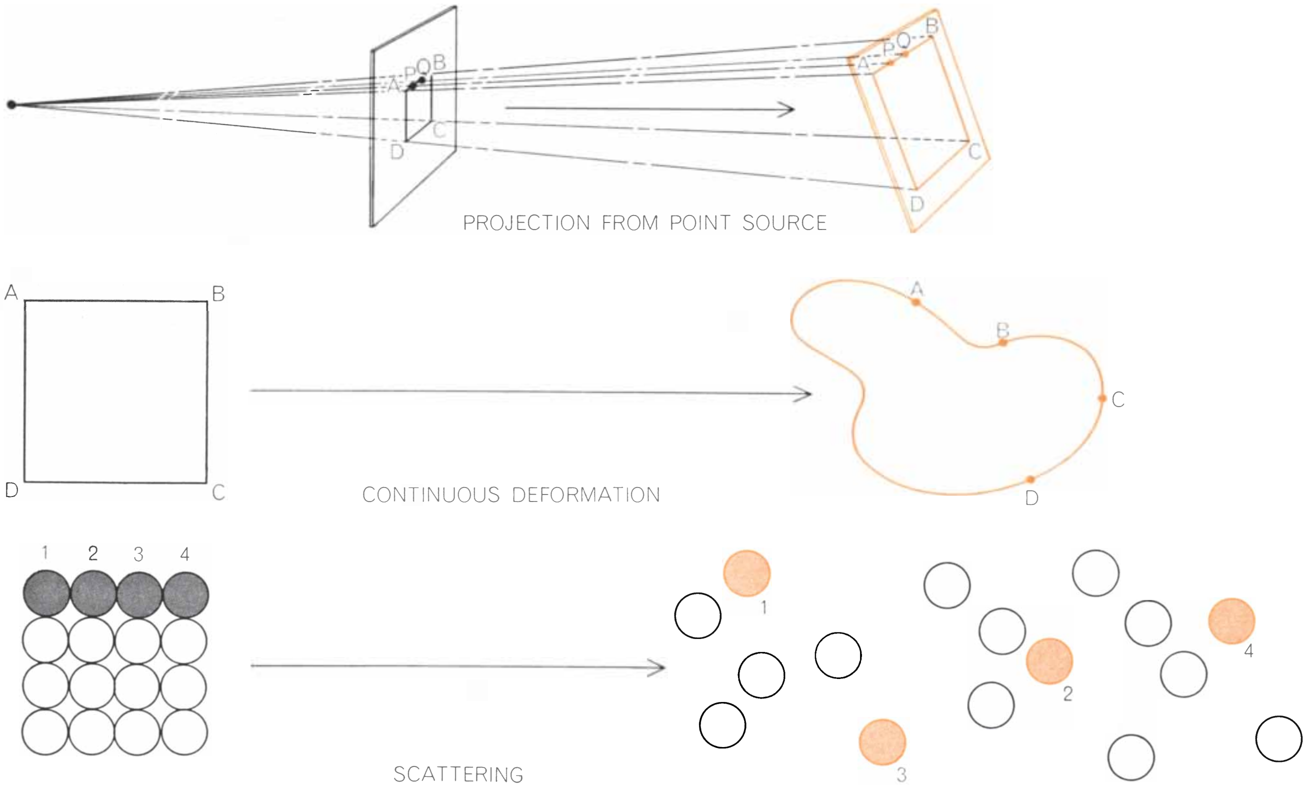
TYPES OF GEOMETRY were classified by Felix Klein according to the invariant properties of figures when they undergo various groups of transformations. Euclidean geometry is represented at top left as the study of properties such as "angle" that are retained when the square *ABCD* is translated, rotated, reflected or dilated. Affine geometry, represented at bottom left, permits all these transformations, and projection by parallel rays to a plane that can be tilted. In this instance the ratio of collinear points is constant. (If *P* is a point on the line *AB*, then the ratio of *AP* to *PB* does not change when the figure is transformed.) At top right is a representation of projective geometry, which permits point-source projection to a randomly tilted screen. An invariant propthe end product of mathematical thought can often be crystallized. It signifies ultimate success in penetrating and ordering mathematical substance and laying bare its skeletal structure. But emphasis on this aspect of mathematics is totally misleading if it suggests that construction, imaginative induction and combination and the elusive mental process called intuition play a secondary role in productive mathematical activity or genuine understanding. In mathematical education, it is true, the deductive method starting from seemingly dogmatic axioms provides a shortcut for covering a large territory. But the constructive Socratic method that proceeds from the particular to the general and eschews dogmatic compulsion leads the way more surely to independent productive thinking.

Just as deduction should be supplemented by intuition, so the impulse to progressive generalization must be tempered and balanced by respect and love for colorful detail. The individual problem should not be degraded to the rank of special illustration of lofty general theories. In fact, general theories emerge from consideration of the specific, and they are meaningless if they do not serve to clarify and order the more particularized substance below.

The interplay between generality and individuality, deduction and construction, logic and imagination—this is the profound essence of live mathematics. Any one or another of these aspects of mathematics can be at the center of a given achievement. In a far-reaching development all of them will be involved. Generally speaking, such a development will start from the "concrete" ground, then discard ballast by abstraction and rise to the lofty layers of thin air where navigation and observation are easy; after this flight comes the crucial test of landing and reaching specific goals in the newly surveyed low plains of individual "reality." In brief, the flight into abstract generality must start from and return again to the concrete and specific.

These principles are dramatically and convincingly illustrated in the evolution of the mathematical sciences. Johannes Kepler, with the genius of the true diagnostician, abstracted from the wealth of Tycho Brahe's observations the elliptical shape of the planetary orbits. Isaac Newton, by further abstraction, derived from these models the universal law of gravitation and the differential equations of mechanics. On this elevated level of unencumbered mathematical abstraction mechanics gained an enormous mobility. On descent to concrete and specific earthbound problems it has won success after success in enormous regions outside its original province of celestial dynamics.

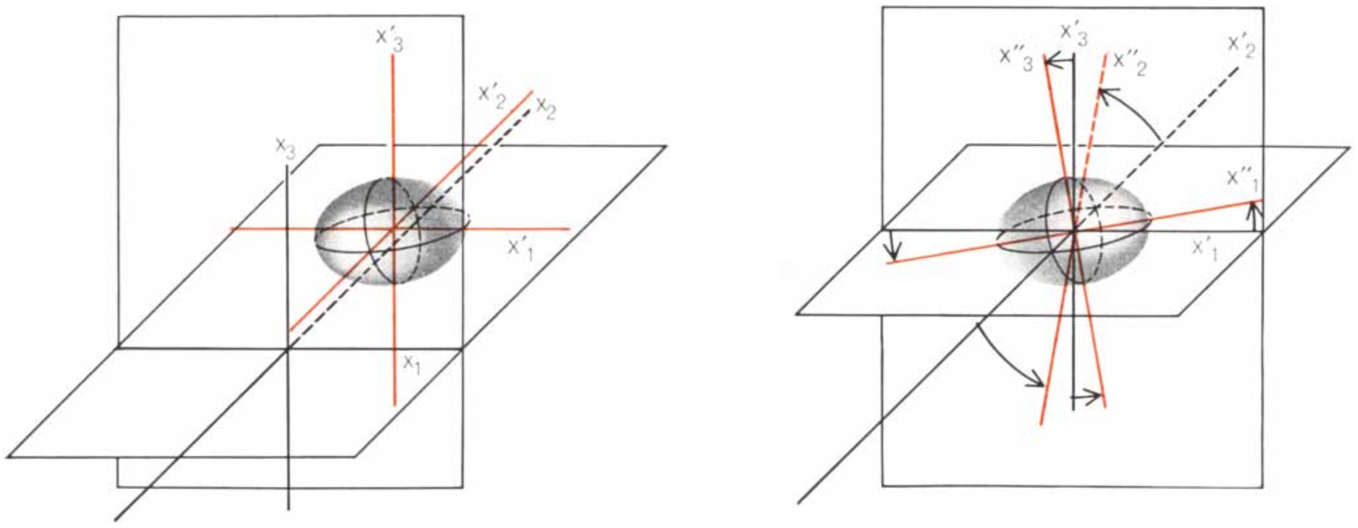
Similarly, in electromagnetism Michael Faraday established a body of experimental findings linked by his own ingenious interpretations. From these some mathematical qualitative laws of electromagnetism were soon abstracted. Then, behind the formulations for specific, simple configurations, the genius of James Clerk Maxwell divined a very general quantitative law that combines in a system of differential equations the magnetic and electric forces and their rates of change. These equations, abstracted and cut loose from specific, tangible cases, may at first have seemed



erty of figures projected this way is the cross ratio of collinear points. (If P and Q are points on the line AB, the ratio of $AP/PQ : AD/PD$ is unchanged by the transformation.) Topology, a fourth type of geometry that is represented at middle right, studies properties preserved during the bending, stretching and twisting operations called continuous deformation. The order of four points A, B, C and D remains after the deformation. In point-set theory, the type of geometry shown at bottom right, the order of points is not retained during the kind of transformation called "scattering." The scattered points *do* remain conumerous with the points in the original figure. Thus point-set theory can be described as the study of the properties preserved under all one-to-one correspondences.

$$25x_1^2 + 22x_2^2 + 16x_3^2 + 20x_1x_2 - 4x_1x_3 - 16x_2x_3 - 62x_1 - 32x_2 - 44x_3 + 55 = 0$$

$$25x_1'^2 + 22x_2'^2 + 16x_3'^2 + 20x_1'x_2' - 4x_1'x_3' - 16x_2'x_3' - 36 = 0$$



$$x_1 = x_1' + 1$$

$$x_2 = x_2' + 1$$

$$x_3 = x_3' + 2$$

$$x_1' = \frac{1}{3}(-x_1'' + 2x_2'' + 2x_3'')$$

$$x_2' = \frac{1}{3}(2x_1'' - x_2'' + 2x_3'')$$

$$x_3' = \frac{1}{3}(2x_1'' + 2x_2'' - x_3'')$$

ALGEBRA AND GEOMETRY of bringing a quadratic surface into normal form is shown for the case of an ellipsoid with center at point (1, 1, 2) of the coordinate system in which it is considered. By parallel translation the coordinate system can be moved to a new position (*colored axes at left*) so the center of the ellipse is at its origin (*0, 0, 0*). The algebra of this translation calls for the substitutions yielding the equation above the middle diagram. The principal axes of the ellipsoid can be made coincident with the translated coordinate system by rotation of its axes to the position given by the colored lines in the middle diagram. Further

too esoteric for application. It soon became clear, however, that Maxwell's ascent to abstraction had opened the way to further progress in a number of directions. The Maxwell equations illuminated the wave nature of electromagnetic phenomena, inspired the experiments of Heinrich Hertz on the propagation of radio waves, started the growth of an entire new technology and led investigators to new lines of research, including, for example, the now very active field of magnetohydrodynamics.

It cannot be said that Maxwell's equations were the product of systematic deductive thinking. Neither should his achievement be ascribed to purely inductive Socratic processes. Instead Maxwell must be counted among those rare minds that recognize similarities and parallels between seemingly remote, disconnected facts and arrive at a major new insight by combining patently diverse elements into a unified system.

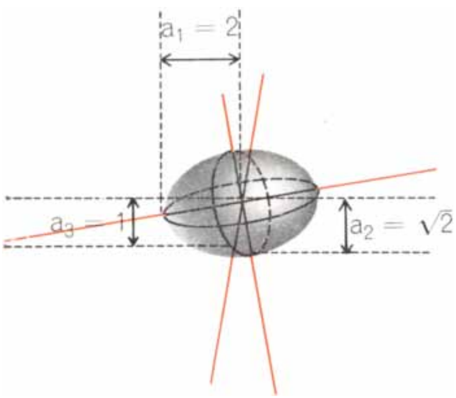
In mathematics proper a corresponding arc of development—from concrete individual substance through abstraction and back again to the concrete and individual—endows a theory with its meaning and significance. To appreciate this basic fact one must bear in mind that the terms "concrete," "abstract," "individual" and "general" have no stable or absolute meaning in mathematics. They refer primarily to a frame of mind, to a state of knowledge and to the character of mathematical substance. What is already absorbed as familiar, for example, is readily taken to be concrete. The words "abstraction" and "generalization" describe not static situations or end results but dynamic processes directed from some concrete stratum to some "higher" one.

Fruitful new discoveries in mathematics sometimes come suddenly with relatively little apparent effort: the view is cleared by abstracting from concrete material and laying bare the structurally essential elements. Axiomatics, irrespective of its Euclidean form, means just that. A recent instance of the fruitful use of abstraction is the generalization by John von Neumann and others of David Hilbert's "spectral" theory, from what proved to be the special case of "bounded" linear operators to "unbounded" ones.

This far-reaching development can be traced in a succession of abstractions upward from the familiar concrete ground of analytic geometry. In the elementary analytic geometry of a three-dimensional space with coordinates x_1, x_2, x_3 a plane is characterized by a linear equation, and a quadratic surface, such as that of a sphere or an ellipsoid, is characterized by a quadratic equation (that is, an equation in which the highest power of an unknown is its square) in the variables x_1, x_2, x_3. For example, an equation of the general form $\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 = 1$ describes a quadratic surface centered at the origin of the coordinate system and with its three principal axes pointing in the direction of the coordinate axes. In the case of the ellipsoid the "coefficients" λ_1, λ_2, λ_3 stand for fixed positive numbers; they represent the expressions $1/a_1^2$, $1/a_2^2$, $1/a_3^2$, in which a_1, a_2, a_3 are the semi-axes of the ellipsoid. The ellipsoid consists precisely of those points for which the values of the variables x_1, x_2, x_3 satisfy the equation [*see illustration on these two pages*].

Now, without much ado, the algebraization of geometry permits one to speak of a space of more than three dimensions, say n dimensions, with coordinates x_1, x_2, $x_3 \ldots x_n$. In this space planes are again defined by linear equations and quadratic surfaces by quad-

$$\frac{1}{4}x_1''^2 + \frac{1}{2}x_2''^2 + x_3''^2 = 1$$



$$\lambda_1 = \frac{1}{a_1^2} = \frac{1}{4}$$

$$\lambda_2 = \frac{1}{a_2^2} = \frac{1}{2}$$

$$\lambda_3 = \frac{1}{a_3^2} = 1$$

substitution yields the equation in normal form, shown at the right above the ellipsoid it describes. The lengths of the semiaxes (a_1, a_2, a_3) are related to the coefficients of the terms in the equation as indicated.

ratic equations in the variables x_1, x_2, $x_3 \ldots x_n$. It is one of the most important results of "linear algebra" that quadratic surfaces can be brought into the algebraic normal form $\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 \ldots \lambda_n x_n^2 = 1$ after a suitable transformation of the coordinate system (or a rigid motion of the figure) that centers the figure at the origin with its principal axes along the coordinate axes. This theorem is the key to many applications; for instance, to the theory of mechanical or electrical systems involving the vibrations of a finite number n of mass points or circuit elements about a state of equilibrium.

Physicists such as Lord Rayleigh did not hesitate to apply this result, without mathematical justification, in a much more general way by letting the number of dimensions n tend to infinity. This step toward greater generalization and abstraction of the underlying mathematics has served well in the study of vibrating systems consisting not of a finite number of mass points or circuit elements but rather of a continuum of matter, such as a string, a membrane or a transmission line.

Hilbert, one of the truly great mathematicians of the past generation, recognized that such quadratic forms of infinitely many variables ought to be secured in a complete mathematical theory. In this endeavor he found it necessary first to restrict the domain of the variables by requiring that the sum of their squares should "converge," that is, have a finite value. Another way of stating this, with the help of a "generalized" Pythagorean theorem, is to say that a point in a "Hilbert space" of infinitely many dimensions must have a finite distance $r = \sqrt{x_1^2 + x_2^2 + \ldots}$ from the origin. Next, Hilbert defined the quadratic form in infinitely many variables—the bounded form—as a double infinite sum of the form

$$a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + \ldots \\ + a_{22}x_2^2 + a_{23}x_2x_3 + \ldots \\ + \ldots,$$

where the first index (that is, x_1 in the first row, x_2 in the second row and so on) goes to infinity from row to row and the second index (that is, x_2 in the first row, x_3 in the second row and so on) goes to infinity along the row. This double infinite sum is under the crucial restriction that it must converge at every point in a Hilbert space.

In such a space many concepts relating to the properties of planes and the quadratic surfaces in finite-dimensional geometry remain meaningful. This is true in particular of the theory of the transformation of quadratic forms to their principal axes. Hilbert showed that every quadratic form in this class can be brought into a normal form by a rotation of the coordinate system. By analogy with the finite-dimensional case Hilbert called the set of values λ_1, λ_2, λ_3 appearing in this normal form the "spectrum" of the quadratic form.

In his generalization of the principal-axis theory from ordinary quadratic forms in n variables to forms in infinitely many variables Hilbert discovered many new phenomena, such as the occurrence of continuous mathematical spectra. Moreover, Hilbert's work served well in the emergence of quantum mechanics. His term "mathematical spectra" found a prophetic relevance to the spectra of energy states in atoms and their constituent particles. But Hilbert's theory of quadratic forms was not quite equal to the task of handling quantum mechanics; the forms occurring there turned out to be "unbounded."

At this point von Neumann, inspired by Erhard Schmidt and more inclined toward abstraction than his elders, carried the process of abstraction another crucial stratum upward. By discarding Hilbert's concept of a quadratic form as something that can be expressed concretely as an infinite algebraic expression and instead formulating the concept abstractly, he was able to avoid its earlier limitations. Thus extended, Hilbert's spectral theory was made to answer the tangibly concrete needs of contemporary physics.

The theory of groups, a central concern of contemporary mathematics, has evolved through an analogous progression of abstractions. Group theory traces its origins back to a problem that has fascinated mathematicians since the Middle Ages: the solving of algebraic equations of degree greater than two by algebraic processes, that is, by addition, subtraction, multiplication, division and extraction of roots. The theory of quadratic equations was known to the Babylonians, and the solution of equations of the third and the fourth degree was accomplished by the Renaissance mathematicians Girolamo Cardano and Niccolò Tartaglia. The solution of equations of the fifth degree and higher degrees, however, encountered insurmountable obstacles.

Early in the 19th century a novel and profound attack on these old problems was launched by Joseph Louis Lagrange, P. Ruffini and Niels Henrik Abel and, in a most original way, by Évariste Galois. These new approaches started from the known facts that an algebraic equation of degree n of the form $x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0$ has n roots r_1, $r_2 \ldots r_n$, and that this set of n roots determines the equation uniquely. (For example, if 1 and 3 are roots of a quadratic equation, then $(x-1)(x-3) = x^2 - 4x + 3 = 0$ is the equation determined by the roots 1 and 3.) The coefficients of the equation are symmetric functions of the roots; that is, they depend on the set of roots regardless of the order. (For example, in a cubic equation $x^3 + ax^2 + bx + c = 0$ with roots r_1, r_2, r_3 the coefficients can be written $-a = r_1 + r_2 + r_3$, $b = r_1r_2 + r_2r_3 + r_3r_1$, $c = r_1r_2r_3$, and if r_1, r_2, r_3 are permuted, a, b and c are not changed.)

Over the years work with such equations revealed that the key to the problem of expressing the roots of the equations in terms of the coefficients lies not only in the study of symmetric expressions but also more decisively in the study of not completely symmetric expressions and in the analysis of whatever symmetries they possess. The expression $E = r_1r_2 + r_3r_4$ does not, for example, remain unchanged for all arbitrary permutations of the four symbols r_1, r_2, r_3, r_4. If the indices 1 and 2 or 3 and 4 are interchanged, E is invariant, that is, remains unchanged. If 1 and 3 are interchanged, however, the resulting expression is not E. On the other hand, the succession of two permutations that changes and then restores E amounts to a permutation that clearly leaves E invariant. The set of these permutations, called a "group" by Galois, represents the intrinsic symmetries of the expression E. The understanding of permutation groups was recognized by the ingenious Galois as the key to a deeper theory of algebraic equations.

Soon afterward mathematicians were discovering permutation groups in other fields. The set of six motions that carry an equilateral triangle into itself, for example, forms a group [*see illustration on page 71*]. Other groups have been uncovered as fundamental structural elements in most of the branches of mathematics.

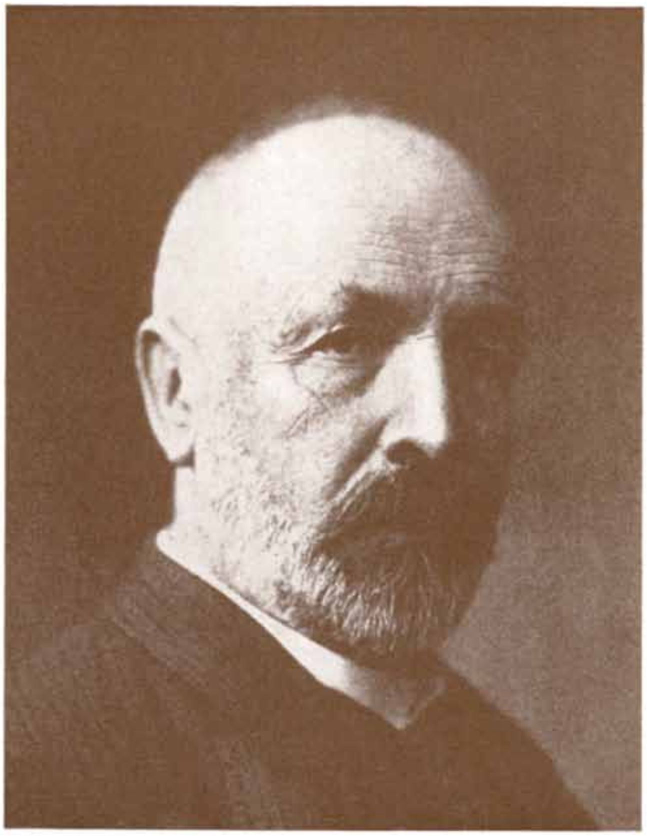
To embrace such groups, in all their different guises and manifestations, in a single concept and to anticipate the even wider scope of undiscovered possibilities required formulation of the underlying group concept in the most abstract terms. This has been done by calling a set of mathematical objects a group if a rule is given for "combining" two elements so as to obtain again an element S of the set; this rule is required to be associative, that is, $(ST)U = S(TU) = S$. Furthermore, the set must include a "unit" element I that, when combined with any other element S of the set, yields S, that is, $IS = SI = S$. Finally, for every element S in the set there must be an "inverse" element S^{-1} such that the combination SS^{-1} yields the unit element, that is, $SS^{-1} = I$.

The specific "substantial" nature of the group is left wide open, of course, by this abstract definition. The elements may be numbers, rotations of geometric bodies, deformations of space (such deformations may be defined by linear or other transformation of the coordinates) or, as above, the permutations of n objects.

Altogether the group concept and the clarification and unification it brought to the diverse branches of mathematics must be reckoned a major achievement of the past 150 years. Much of the effort has been expended on the intermediate, lofty sector of the arc of development, that is, on the structural analysis of the abstracted concepts. The work has contributed all along, however, to illuminating more specific concrete areas, such as number theory and algebra. One of the remarkable successes along this line was Felix Klein's famous classification in the 1870's of the various branches of geometry according to groups of transformations under which certain geometrical properties remain invariant [*see illustration on pages 42 and 43*].

Abstract group theory has found significant application in the still more concrete problems of particle physics. Here the occasion is provided by the intricate group of open and hidden symmetries that prevail in the configuration and interaction of the nuclear particles. The success of group theory in bringing order to a great mass of data and predicting the existence of new particles [see "Mathematics in the Physical Sciences," page 128] shows convincingly how abstraction can help in the search for hard facts.

Intuition, that elusive vital agent, is always at work in creative mathematics, motivating and guiding even the most



INFLUENTIAL MATHEMATICIANS who helped to direct the course of 20th-century thought are shown on these two pages. Georg Cantor (*left*) suggested an order for infinite sets, thus focusing mathematical speculation on set theory. Henri Poincaré (*middle*)

abstract thinking. In its most familiar manifestation, geometrical intuition, it has figured in the many major recent advances in mathematics that have occurred in or flowed from work in geometry. Yet there is a powerful compulsion in mathematics to reduce the visible role of intuition, or perhaps one may better say to buttress it, by precise and rigorous reasoning.

Topology, the youngest and most vigorous branch of geometry, illustrates in a spectacular way the fruitful working of this tension between intuition and reason. With a few isolated but important earlier discoveries—for example the one-sided Möbius band—as its stock-in-trade, topology emerged as a field of serious study in the 19th century. For a long period it was almost entirely a matter of geometrical intuition, of cutting and pasting together surfaces in an effort to visualize the mathematical substance of topology, that is, the properties of surfaces that do not change under arbitrary continuous deformation. Early in the evolution of the new discipline, however, Georg Friedrich Bernhard Riemann brought it to the center of attention. In his sensational work on the theory of algebraic functions of a complex variable (a variable incorporating the imaginary number $\sqrt{-1}$) he showed that the topological facts concerning what are now called Riemann surfaces are essential to a real understanding of these functions.

During the 19th century investigators discovered and systematically explored a wide range of topological properties of surfaces of two, three and then of *n* dimensions. Still on a more or less intuitive basis, early in this century, the great Henri Poincaré and others built a fascinating edifice of topological theory. This work proceeded in close relation to the development of group theory and found uses in other fields of mathematics and in the evolution of the mathematical sciences to higher levels of sophistication. It was put to work, for example, in celestial mechanics, specifically in the construction of planetary orbits in space curved by gravitational fields.

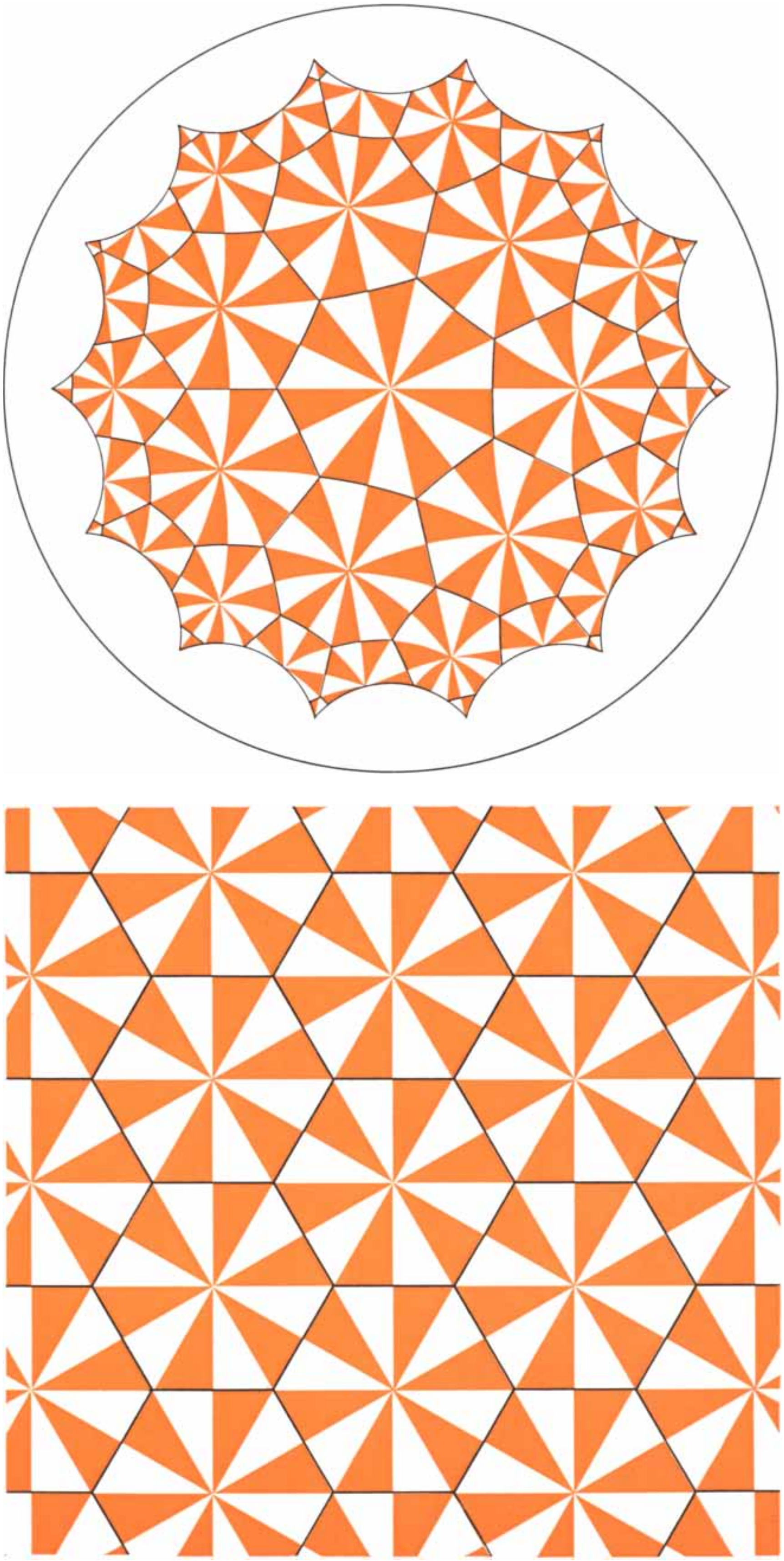
Topologists soon began to feel with urgency the need to sharpen their tools in order to catch the products of geometrical intuition in the vise of modern mathematical precision—without destroying their convincing beauty. This task was accomplished almost single-handed in the first decades of this century by the Dutch mathematician L. E. J. Brouwer. Thanks to his gigantic effort, topology is now as amenable to rigorous treatment as the geometry of Euclid, and advances in the field proceed on the solid ground of logically impeccable reasoning.

warned against preoccupation with set theory. David Hilbert (*right*) generalized the principal-axis theory. In 1900 he proposed 23 projects for 20th-century mathematicians.

At the center of the difficulties confronting Brouwer was the dilemma presented by the concept of continuity. Everyone has a sure intuitive idea of what continuity is, for example the smoothness of a curve. But the beginning student of calculus loses his assurance at the very outset as he attempts to capture continuity in a precise mathematical formulation. Difficulty is inherent in the task because the geometrical intuition of continuity and the mathematical logical concept do not perfectly match. Rigorous definition brings to the surface whole areas of cases, perhaps marginal, that confound the intuition with paradox. It is easy to construct, for instance, continuous curves (in the exact sense of the definition) that do not have a length [*see lower illustration on page 49*], that have nowhere a direction or that wind around, without any self-intersection, within a square so that they come arbitrarily close to any given point in the square. Such bizarre constructions highlight the need for careful reasoning in proofs of the topological properties of surfaces or other objects subjected to complex continuous deformations.

That need is not at once intuitively apparent to the nontopologist. Consider, for example, C. Jordan's famous theorem stating that any nonintersecting continuous closed curve in a plane bounds two separate domains—the interior and the exterior [*see upper illustration on page 49*]. Every scientist, engineer and student in his naïve right mind will regard the effort to prove such a theorem as an unnecessary, self-imposed, almost masochistic exercise. Yet in writing his classical textbook on analysis Jordan felt strongly the need for a proof and presented one. It is a measure of the subtlety of the problem that Jordan's proof turned out to be not completely correct! Similarly, no one will doubt that the dimensionality of a two-dimensional or three-dimensional geometric figure remains unchanged under any continuous deformation. Yet the precise proof of this fact, under the general assumption of mere abstract continuity, stands as one of Brouwer's major achievements.

It is possible, of course, to evade some



MAPPING A FUNCTION of a complex variable from an infinitely many-sheeted Riemann surface produces the figure shown at top of this illustration. The circle-arc polygons that grow infinitely small toward the outer circle correspond to straight-line polygons (*bottom*) that extend infinitely without changing size throughout the plane on which they are shown.

of the difficulties in the notion of continuity by restricting the group of continuous deformations—by demanding, for example, "smoothness" or differentiability instead of pure continuity. This has been done with great success. Differential topology, as it is called, has recently achieved outstanding results. Investigation of deformations conducted under the requirement of "reasonable" smoothness has produced significantly different classifications of topological structures than would be yielded under a regime of completely general continuity.

These developments may also be welcomed as indicating a healthy deflection of the trend toward boundless generality. Ever since Georg Cantor's achievements in the theory of sets, in the last decades of the 19th century, that trend has occupied many mathematical minds. Some great mathematicians, notably Poincaré, have fought it bitterly as a menace to mathematics, in particular because it leads to unresolved paradoxes. If Poincaré's militant criticism has proved to be overly restrictive and even reactionary, it was nonetheless salutary because it encouraged constructive mathematicians concerned with specific and graspable matters.

Various motivations, in the same individual or in different people, inspire mathematical activity. Certainly the roots in physical reality of large parts of mathematics—especially analysis—supply powerful motivation and inspiration. The situation with respect to other realms of reality is not much different. In number theory and algebra it is the intriguing reality of the world of numbers, so deeply inherent in the human mind. Still more removed from physical reality, one might think, is the reality of the logical processes involved in mathematical thinking. Yet basic ideas from esoteric work in mathematical logic have proved useful for the understanding and even for the design of automatic computing machines.

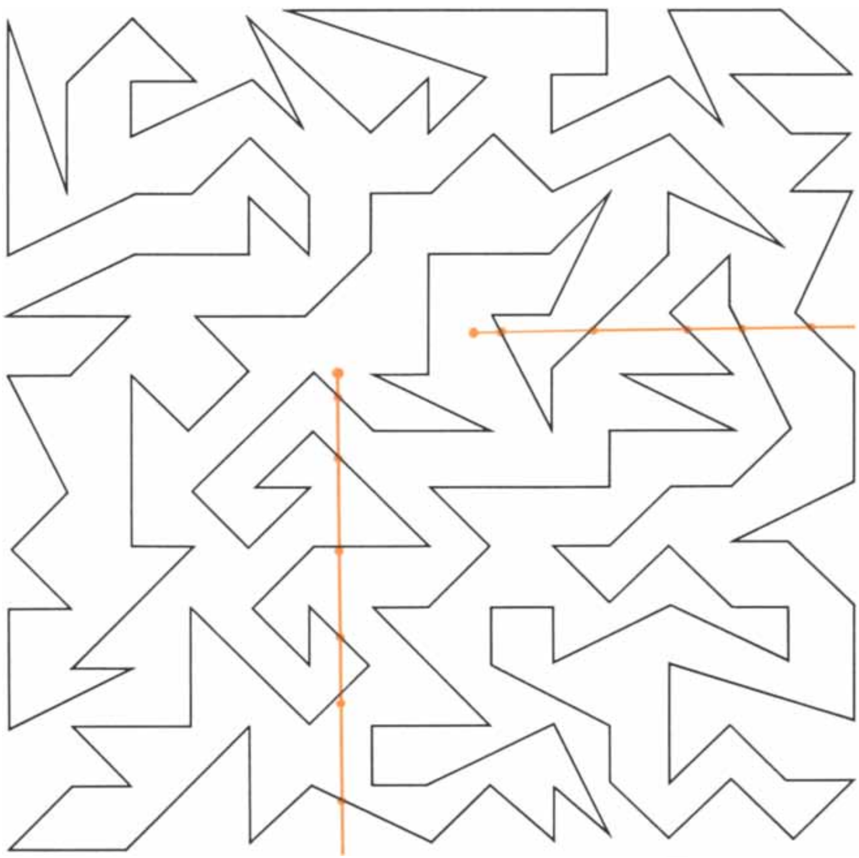
In brief, mathematics must take its motivation from concrete specific substance and aim again at some layer of "reality." The flight into abstraction must be something more than a mere escape; start from the ground and reentry are both indispensable, even if the same pilot cannot always handle all phases of the trajectory. The substance of the purest mathematical enterprise may often be provided by tangible physical reality. That mathematics, an emanation of the human mind, should serve so effectively for the description and understanding of the physical world is a challenging fact that has rightly attracted the concern of philosophers. Leaving philosophical questions aside, however, the engagement in physical questions or the apparent absence of such engagement must not be taken as a criterion for distinguishing between the kinds of mathematics and mathematicians.

No sharp dividing line can, in fact, be drawn between "pure" and "applied" mathematics. There should not be a class of high priests of unadulterated mathematical beauty, exclusively responsible to their own inclinations, and a class of workers who serve other masters. Class distinctions of this kind are at best the symptom of human limitations that keep most individuals from roaming at will over broad fields of interest.

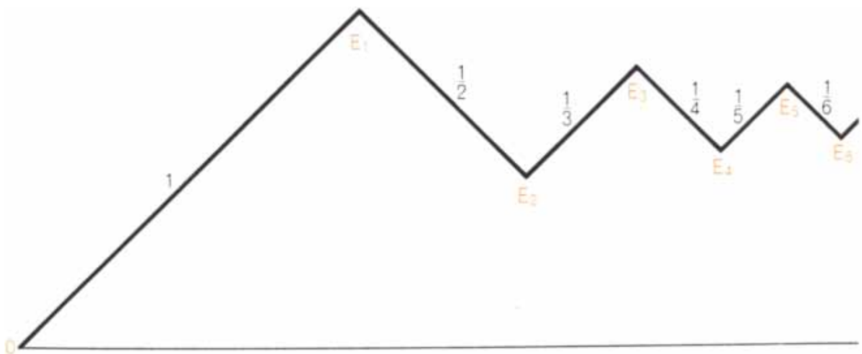
Although the substance of mathematics is indivisible, distinct differences must be acknowledged in the attitudes that the same scientist or different scientists may bring to a problem. The attitude of the purist, which every scientifically inclined mind will at least sometimes assume, demands uncompromising perfection. No gaps or rough spots can be tolerated in the solution of a problem and the result must flow from an unbroken chain of flawless reasoning. If the attempt encounters insurmountable obstacles, then the purist is inclined to restate his problem or replace it with another in which the difficulty is capable of being managed. He may even "solve" his problem by redefining what he means by a solution; this is, in fact, a not uncommon preliminary step toward a true solution of the original problem.

In the case of applied research the situation is different. The problem, in the first place, cannot be as freely modified or avoided; what is wanted is a believable, humanly reliable answer. If necessary, therefore, the mathematician must accept compromise; he must be willing to interpolate guesswork into the train of reasoning and to make allowance for the uncertainty of numerical evidence. But even the most practically motivated study—the analysis, for instance, of flow involving shock discontinuities—may demand fundamental mathematical investigation to discover how to frame the question. Pure existence proofs may also be significant in applied research; ascertaining that a solution exists may give the needed assurance of the suitability of the mathematical model. Finally, applied mathematics is dominated by approximations; these are inescapable in the attempt to mirror physical processes in mathematical models.

To handle the translation of reality into the abstract models of mathematics and to appraise the degree of accuracy thereby attainable calls for intuitive feeling sharpened by experience. It may also often involve the framing of genuine mathematical problems that are far too difficult to be solved by the available techniques of the science. Such, in part, is the nature of the intellectual adventure and the satisfaction experienced by the mathematician who works with engineers and natural scientists on the mastering of the "real" problems that arise in so many places as man extends his understanding and control of nature.



JORDAN CURVE THEOREM states that any closed curve such as the one shown here bounds interior and exterior domains. A line drawn from inside the curve to the outside will make an odd number of intersections; a line drawn from outside it, an even number.



INFINITE ZIGZAG is composed of successive segments with lengths 1, ½, ⅓, ¼, ⅕, ⅙. . . . The sequence of unit fractions has no finite sum and the curve itself has no finite length.

NUMBER

With geometry it is one of the two pillars at the base of mathematics. The concept of number is enlarged by building up number systems and by seeking to break them down into their most primitive elements

by Philip J. Davis

By popular definition a mathematician is a fellow who is good at numbers. Most mathematicians demur. They point out that they have as much difficulty as anybody else in reconciling their bank statements, and they like to refer to supporting anecdotes, such as that Isaac Newton, who was Master of the Mint, employed a bookkeeper to do his sums. They observe further that slide rules and electronic computers were developed as crutches to help mathematicians.

All of this is obviously irrelevant. Who, if not the mathematician, is the custodian of the odd numbers and the even numbers, the square numbers and the round numbers? To what other authority shall we look for information and help on Fibonacci numbers, Liouville numbers, hypercomplex numbers and transfinite numbers? Let us make no mistake about it: mathematics is and always has been the numbers game par excellence. The great American mathematician G. D. Birkhoff once remarked that simple conundrums raised about the integers have been a source of revitalization for mathematics over the centuries.

NEW-WORLD NUMBERS, in dot-and-bar notation, record the date of a fragmentary Olmec stela from the state of Vera Cruz in Mexico. Each dot equals one unit; each bar equals five. Restored, these numbers show seven periods of 400 "years" (*missing from top*), plus 16 periods of 20 "years" (*the topmost surviving numeral, dot eroded*), plus six "years" of 360 days each, plus 16 "months" of 20 days each, plus 18 days: a total elapsed time of nearly 3,127 "years" since the mythical start of the system. By one method of correlation with the Christian calendar, this is the equivalent of November 4, 291 B.C., and is the second oldest recorded date in the Western Hemisphere.

Numbers are an indispensable tool of civilization, serving to whip its activities into some sort of order. In their most primitive application they serve as identification tags: telephone numbers, car licenses, ZIP-code numbers. At this level we merely compare one number with another; the numbers are not subjected to arithmetical operations. (We would not expect to arrive at anything significant by adding the number of Leonard Bernstein's telephone to Elizabeth Taylor's.) At a somewhat higher level we make use of the natural order of the positive integers: in taking a number for our turn at the meat counter or in listing the order of finish in a race. There is still no need to operate on the numbers; all we are interested in is whether one number is greater or less than another. Arithmetic in its full sense does not become relevant until the stage at which we ask the question: How many? It is then that we must face up to the complexities of addition, subtraction, multiplication, division, square roots and the more elaborate dealings with numbers.

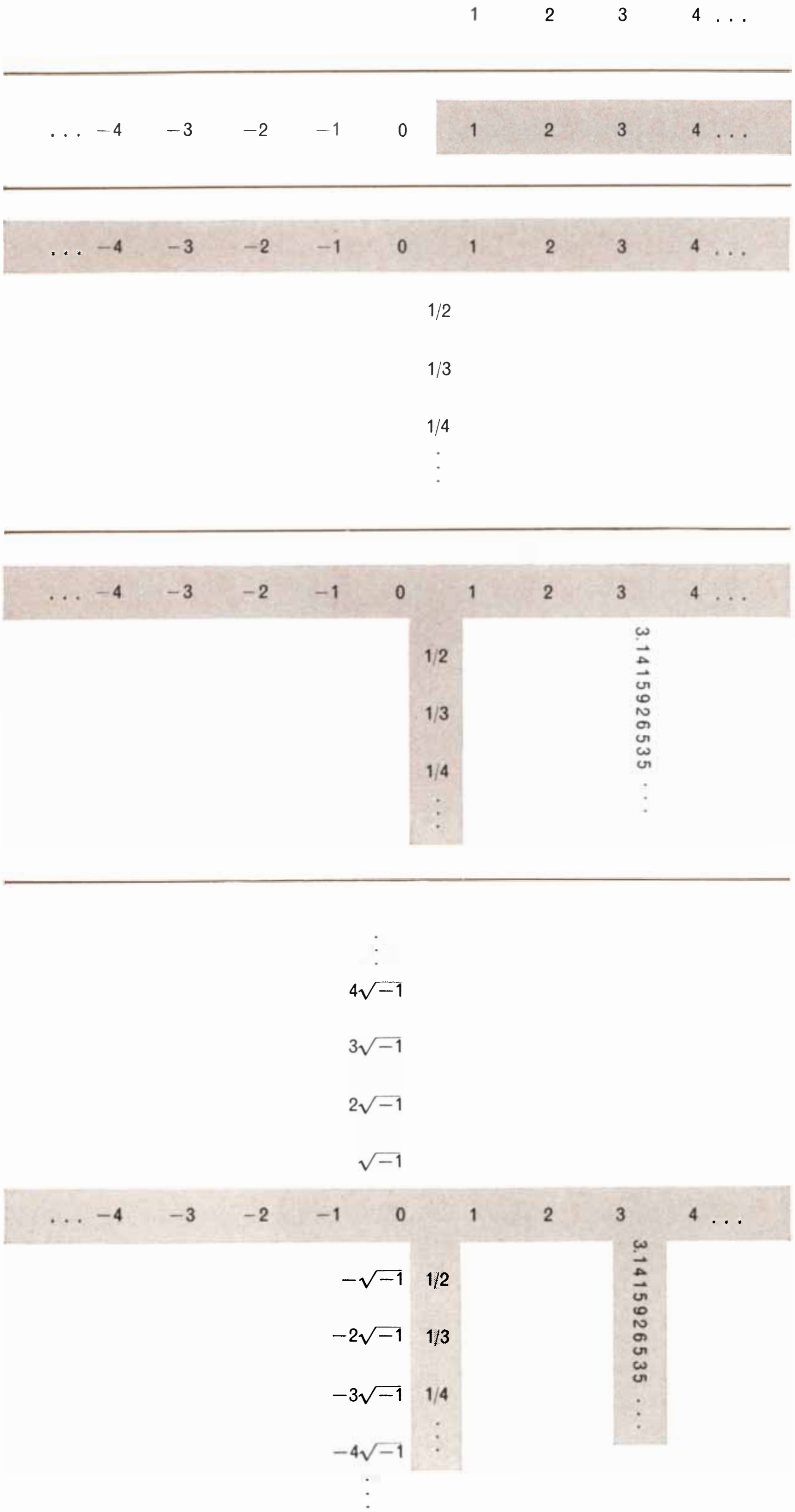
The complexity of a civilization is mirrored in the complexity of its numbers. Twenty-five hundred years ago the Babylonians used simple integers to deal with the ownership of a few sheep and simple arithmetic to record the motions of the planets. Today mathematical economists use matrix algebra to describe the interconnections of hundreds of industries [see "Mathematics in the Social Sciences," page 168], and physicists use transformations in "Hilbert space"—a number concept seven levels of abstraction higher than the positive integers—to predict quantum phenomena [see "Mathematics in the Physical Sciences," page 128].

The number systems employed in mathematics can be divided into five principal stages, going from the simplest to the most complicated. They are: (1) the system consisting of the positive integers only; (2) the next higher stage, comprising the positive and negative integers and zero; (3) the rational numbers, which include fractions as well as the integers; (4) the real numbers, which include the irrational numbers, such as π; (5) the complex numbers, which introduce the "imaginary" number $\sqrt{-1}$.

The positive integers are the numbers a child learns in counting. They are usually written 1, 2, 3, 4..., but they can and have been written in many other ways. The Romans wrote them I, II, III, IV...; the Greeks wrote them α, β, γ, δ...; in the binary number system, containing only the digits 0 and 1, the corresponding numbers are written as 1, 10, 11, 100.... All these variations come to the same thing: they use different symbols for entities whose meaning and order are uniformly understood.

Early man needed only the first few integers, but with the coming of civilization he had to invent higher and higher numbers. This advance did not come readily. As Bernard Shaw remarked in *Man and Superman:* "To the Bushman who cannot count further than his fingers, eleven is an incalculable myriad." As late as the third century B.C. there appears to have been no systematic way of expressing large numbers. Archimedes then suggested a cumbersome method of naming them in his work *The Sand Reckoner*.

Yet while struggling with the names of large numbers the Greek mathematicians took the jump from the finite to



NUMBER CONCEPTS can be arrayed in such a way that each succeeding system embraces all its predecessors. The most primitive concept, consisting of the positive integers alone, is succeeded by one extended to include zero and the negative integers. The next two additions are the rational and the irrational numbers, the latter recognizable by their infinitely nonrepetitive sequence of integers after the decimal. This completes the system of real numbers. The final array represents complex numbers, which began as a Renaissance flight of fancy and have since proved vital to the mathematics of physics and engineering. The complex numbers consist of real numbers combined with the quantity $\sqrt{-1}$, or i.

the infinite. The jump is signified by the three little dots placed after the 4 in the series above. They indicate that there is an integer after 4 and another after the successor to 4 and so on through an unlimited number of integers. For the ancients this concept was a supreme act of the imagination, because it ran counter to all physical experience and to a philosophical belief that the universe must be finite. The bold notion of infinity opened up vast possibilities for mathematics, and it also created paradoxes. Its meaning has not been fully plumbed to this day.

Oddly the step from the positive to the negative integers proved to be a more difficult one to make. Negative numbers seem altogether commonplace in our day, when 10 degrees below zero is a universally understood quantity and the youngest child is familiar with the countdown: "...five, four, three, two, one...." But the Greeks dealt with negative numbers only in terms of algebraic expressions of the areas of squares and rectangles, for example $(a - b)^2 = a^2 - 2ab + b^2$ [*see illustration at top of pages 54 and 55*]. Negative numbers were not fully incorporated into mathematics until the publication of Girolamo Cardano's *Ars Magna* in 1545.

Fractions, or rational numbers (the name they go by in number theory), are more ancient than the negative numbers. They appear in the earliest mathematical writings and were discussed at some length as early as 1550 B.C. in the Rhind Papyrus of Egypt. The present way of writing fractions (for instance 1/4, 1/5, 8/13) and also the present way of doing arithmetic with them date from the 15th and 16th centuries. Today most people probably could not be trusted to add 1/4 and 1/5 correctly. (Indeed, how often do they need to?) The handling of fractions, however, is by no means a dead issue. It recently became a matter of newspaper controversy as a result of the treatment of fractions in some of the new school mathematics courses, with the cancellation school pitted against the anticancellation school. The controversy stemmed from a divergence of opinion as to what the practical and aesthetic goals of school mathematics should be; the mystified layman, reading about it over his eggs and coffee, may have been left with the impression that everything he had once been taught about fractions was wrong or immoral.

The irrational numbers also have a long history. In the sixth century B.C. the mathematical school of Pythag-

ABACUS PRINCIPLE

EGYPTIAN		I	II	III	IIII	III/II	III/III	IIII/III	IIII/IIII	III/III/III	∩	∩I	∩II	∩III	∩IIII	III/∩II	III/∩III
MAYAN		•	••	•••	••••	—	•/—	••/—	•••/—	••••/—	=	•/=	••/=	•••/=	••••/=	≡	•/≡
GREEK		A	B	Γ	Δ	E	F	Z	H	Θ	I	IA	IB	IΓ	IΔ	IE	IF
ROMAN		I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI
ARABIC	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
BINARY	00000	00001	00010	00011	00100	00101	00110	00111	01000	01001	01010	01011	01100	01101	01110	01111	10000

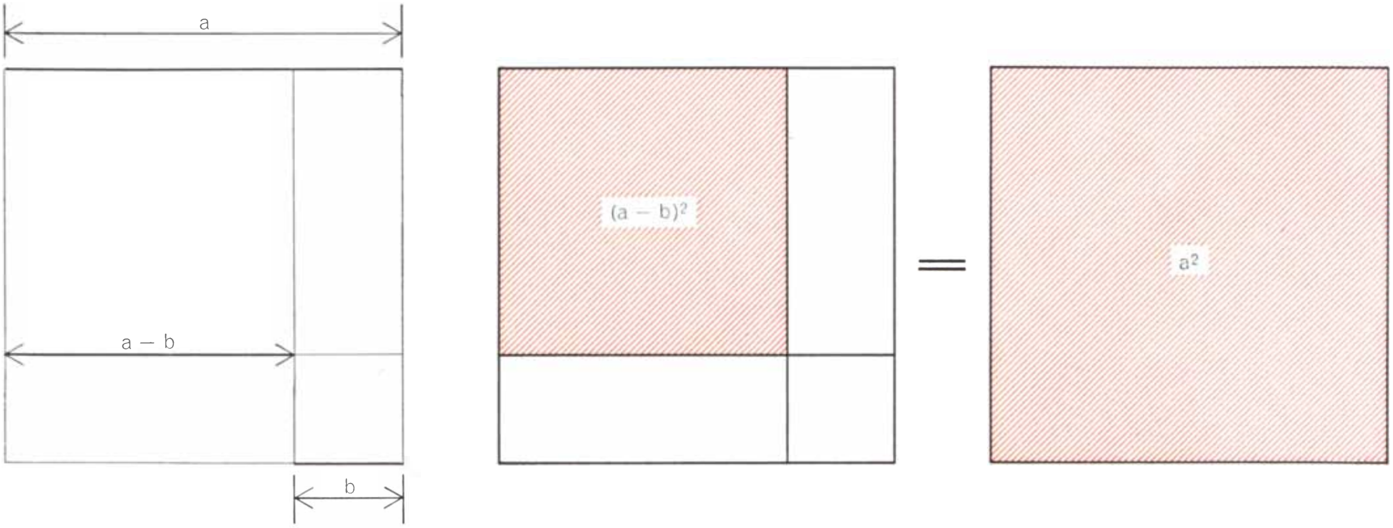
ANCIENT AND MODERN NOTATIONS for the numerals from 1 to 16 are arrayed beneath the equivalent values set up on a two-rod abacus. Of the six examples all but two have a base of 10; these are repetitive above that number regardless of whether the symbol is tally-like or unique for each value. The Mayan notation has a base of 20 and is repetitive after the numeral 5. The binary system has a base of 2 and all its numbers are written with only a pair of symbols, 0 and 1. Thus two, or 2^1, is written 10; four, or 2^2, is written 100, and eight, or 2^3, is written 1000. Each additional power of 2 thereafter adds one more digit to the binary notation.

oras encountered a number that could not be fitted into the category of either integers or fractions. This number, arrived at by the Pythagorean theorem, was $\sqrt{2}$: the length of the diagonal of a square (or the hypotenuse of a right triangle) whose sides are one unit long. The Greeks were greatly upset to find that $\sqrt{2}$ could not be expressed in terms of any number a/b in which a and b were integers, that is, any rational number. Since they originally thought the only numbers were rational numbers, this discovery was tantamount to finding that the diagonal of a square did not have a mathematical length! The Greeks resolved this paradox by thinking of numbers as lengths. This led to a program that inhibited the proper development of arithmetic and algebra, and Greek mathematics ran itself into a stone wall.

It took centuries of development and sophistication in mathematics to realize that the square root of two can be represented by putting three dots after the last calculated digit. Today we press the square-root button of a desk calculator and get the answer: $\sqrt{2} = 1.41421\ldots.$ Electronic computers have carried the specification of the digits out to thousands of decimal places. Any number that can be written in this form—with one or more integers to the left of a decimal point and an infinite sequence of integers to the right of the point—is a "real" number. We can express in this way the positive integers (for example, $17 = 17.0000\ldots$), the negative integers ($-3 = -3.0000\ldots$) or the rational numbers ($17\frac{1}{5} = 17.20000\ldots$). Some rational numbers do not resolve themselves into a string of zeros at the right; for instance, the decimal expression of one-seventh is $1/7 = 0.142857\ 142857\ 142857\ldots.$ What makes these numbers "rational" is the fact that they contain a pattern of digits to the right of the decimal point that repeats itself over and over. The numbers called "irrational" are those that, like the square root of two, have an infinitely nonrepeating sequence of decimal digits. The best-known examples of irrationals are: $\sqrt{2} = 1.4142135623\ldots$ and $\pi = 3.1415926535\ldots.$ The irrational numbers are of course included among the real numbers.

It is in the domain of the "complex numbers" that we come to the numbers called "imaginary"—a term that today is a quaint relic of a more naïve, swashbuckling era in arithmetic. Complex numbers feature the "quantity" $\sqrt{-1}$, which, when multiplied by itself, produces -1. Since this defies the basic rule that the multiplication of two positive or negative numbers is positive, $\sqrt{-1}$ (or i, as it is usually written) is indeed an oddity: a number that cannot be called either positive or negative. "The imaginary numbers," wrote Gottfried Wilhelm von Leibniz in 1702, "are a wonderful flight of God's Spirit; they are almost an amphibian between being and not being."

From Renaissance times on, although mathematicians could not say what these fascinating imaginaries were, they used complex numbers (which have the general form $a + b\sqrt{-1}$) to solve equations and uncovered many beautiful identities. Abraham de Moivre discovered the formula $(\cos\theta + \sqrt{-1}\sin\theta)^n$



NEGATIVE NUMBERS were visualized by the Greeks in terms of lines and bounded areas. Thus they realized that the square erected on the line $a - b$ was equal in area, after a series of manipulations, to the square on the entire line a. The first manipula-

$= \cos\ n\theta + \sqrt{-1}\ \sin\ n\theta$. Leonhard Euler discovered the related formula

$$e^{\pi\sqrt{-1}} = -1$$

(e being the base of the "natural logarithms," 2.71828...).

The complex numbers remained on the purely manipulative level until the 19th century, when mathematicians began to find concrete meanings for them. Caspar Wessel of Norway discovered a way to represent them geometrically [*see illustration on page 56*], and this became the basis of a structure of great beauty known as the theory of functions of a complex variable. Later the Irish mathematician William Rowan Hamilton developed an algebraic interpretation of complex numbers that represented each complex number by a pair of ordinary numbers. This idea helped to provide a foundation for the development of an axiomatic approach to algebra.

Meanwhile physicists found complex numbers useful in describing various physical phenomena. Such numbers began to enter into equations of electrostatics, hydrodynamics, aerodynamics, alternating-current electricity, diverse other forms of vibrating systems and eventually quantum mechanics. Today many of the productions of theoretical physics and engineering are written in the language of the complex-number system.

In the 19th century mathematicians invented several new number systems. Of these modern systems three are particularly noteworthy: quaternions, matrices and transfinite numbers.

Quaternions were Hamilton's great creation. For many years he brooded over the fact that the multiplication of complex numbers has a simple interpretation as the rotation of a plane. Could this idea be generalized? Would it be possible to invent a new kind of number and to define a new kind of multiplication such that a rotation of three-dimensional space would have a simple interpretation in terms of the multiplication? Hamilton called such a number a triplet; just as Wessel represented complex numbers by a point on a two-dimensional plane, the triplets were to be represented by a point in three-dimensional space.

The problem was a hard nut to crack. It was continually on Hamilton's mind, and his family worried over it with him. As he himself related, when he came down to breakfast one of his sons would ask: "Well, Papa, can you multiply triplets?" And Papa would answer dejectedly: "No, I can only add and subtract them."

One day in 1843, while he was walking with his wife along a canal in Dublin, Hamilton suddenly conceived a way to multiply triplets. He was so elated that he took out a penknife then and there and carved on Brougham Bridge the key to the problem, which certainly must have mystified passersby who read it: "$i^2 = j^2 = k^2 = ijk = -1$."

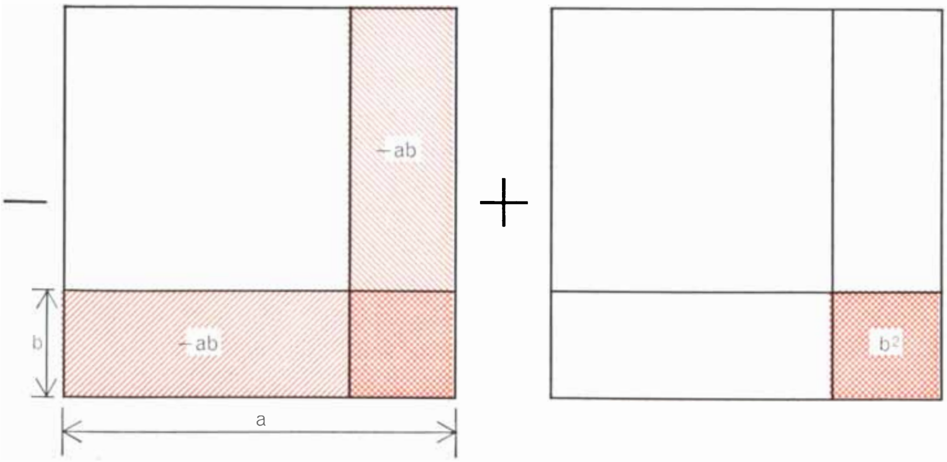
The letters i, j and k represent hypercomplex numbers Hamilton called quaternions (the general form of a quaternion being $a + bi + cj + dk$, with a, b, c and d denoting real numbers). Just as the square of $\sqrt{-1}$ is -1, so $i^2 = -1$, $j^2 = -1$ and $k^2 = -1$. The key to the multiplication of quaternions, however, is that the commutative law does not hold [*see table on page 57*]. Whereas in the case of ordinary numbers $ab = ba$, when quaternions are reversed, the product may be changed: for example, $ij = k$ but $ji = -k$.

The second modern number concept mentioned above, that of the matrix, was developed more or less simultaneously by Hamilton and the British mathematicians J. J. Sylvester and Arthur Cayley. A matrix can be regarded as a rectangular array of numbers. For example,

$$\begin{pmatrix} 1 & 6 & 7 \\ 2 & 0 & 4 \end{pmatrix}$$

is a matrix. The entire array is thought of as an entity in its own right. Under the proper circumstances it is possible to define operations of addition, subtraction, multiplication and division for such entities. The result is a system of objects whose behavior is somewhat reminiscent of ordinary numbers and which is of great utility in many provinces of pure and applied mathematics.

The third modern concept, that of transfinite numbers, represents a totally different order of idea. It is entertainingly illustrated by a fantasy, attributed to the noted German mathematician David Hilbert and known as "Hilbert's Hotel." It would be appreciated by roomless visitors to the New York World's Fair. A guest comes to Hilbert's Hotel and asks for a room. "Hm," says the manager. "We are all booked up, but that's not an unsolvable problem



tions require the subtraction of two rectangles of length a and width b from a^2. But these rectangles overlap, and one quantity has been subtracted twice. This is b^2, which is restored.

here; we can make space for you." He puts the new guest in room 1, moves the occupant of room 1 to room 2, the occupant of room 2 to room 3 and so on. The occupant of room N goes into room $N + 1$. The hotel simply has an infinite number of rooms.

How, then, can the manager say that the hotel is "all booked up?" Galileo noted a similar paradox. Every integer can be squared, and from this we might conclude that there are as many squares as there are integers. But how can this be, in view of the known fact that there are integers that are not squares, for instance 2, 3, 5, 6. . . ?

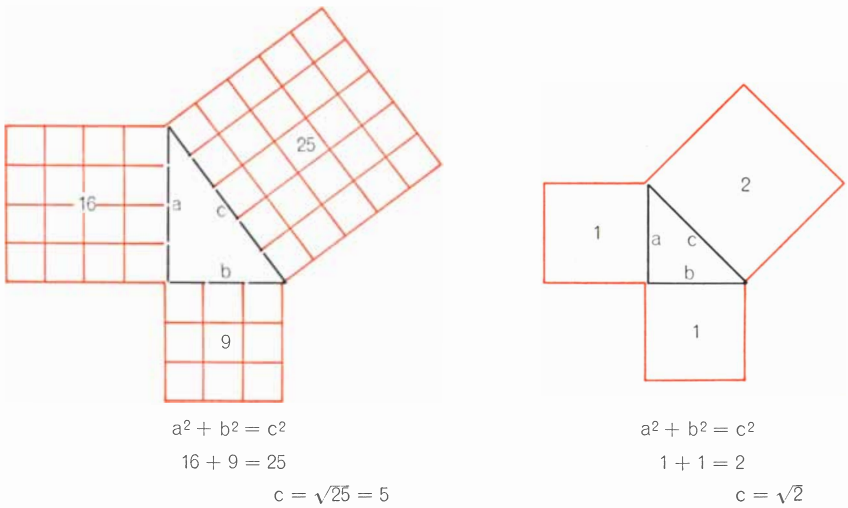
One of the endlessly alluring aspects of mathematics is that its thorniest paradoxes have a way of blooming into beautiful theories. The 19th-century German mathematician Georg Cantor turned this paradox into a new number system and an arithmetic of infinite numbers.

He started by defining an infinite set as one that can be put into a one-to-one correspondence with a part of itself, just as the integers are in a one-to-one correspondence with their squares. He noted that every set that can be put into such correspondence with the set of all the integers must contain an infinite number of elements, and he designated this "number" as $\aleph$ (aleph, the first letter of the Hebrew alphabet). Cantor gave this "first transfinite cardinal" the subscript zero. He then went on to show that there is an infinity of other sets (for example the set of real numbers) that cannot be put into a one-to-one correspondence with the positive integers because they are larger than that set. Their sizes are represented by other transfinite cardinal numbers ($\aleph_1$, $\aleph_2$ and so on). From such raw materials Cantor developed an arithmetic covering both ordinary and transfinite numbers. In this arithmetic some of the ordinary rules are rejected, and we get strange equations such as $\aleph_0 + 1 = \aleph_0$. This expresses, in symbolic form, the hotel paradox.

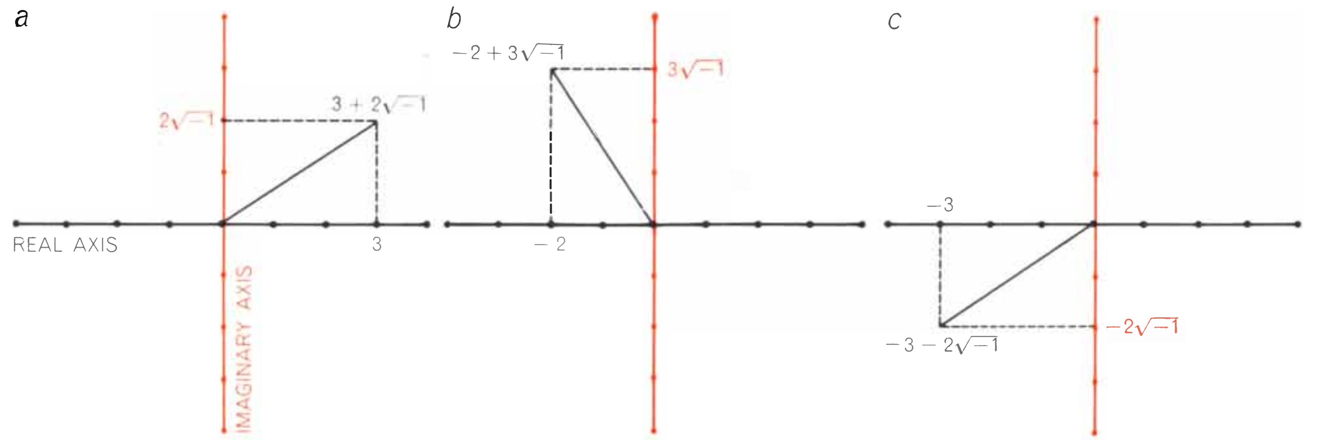
The transfinite numbers have not yet found application outside mathematics itself. But within mathematics they have had considerable influence and have evoked much logical and philosophical speculation. Cantor's famous "continuum hypothesis" produced a legacy of unsolved problems that still occupy mathematicians. In recent years solutions to some of these problems have been achieved by Alfred Tarski of the University of California at Berkeley and Paul J. Cohen of Stanford University.

We have reviewed the subject matter (or dramatis personae) of the numbers game; it now behooves us to examine the rules of the game. To nonmathematicians this may seem to be an exercise in laboring the obvious. The geometry of Euclid is built on "self-evident" axioms, but rigorous examination of the axioms in the 19th century disclosed loopholes, inconsistencies and weaknesses that had to be repaired in order to place geometry on firmer foundations. But, one may ask, what is there about the simple rules of arithmetic and algebra that needs examination or proof? Shaken by the discoveries of the shortcomings of Euclid's axioms, and spurred by the surprising features of the new number concepts such as the quaternions, many mathematicians of the 19th century subjected the axioms of number theory to systematic study.

Are the laws of arithmetic independent, or can one be derived logically from another? Are they really fundamental, or could they be reduced to a more primitive, simpler and more elegant set of laws? Answers to questions such as these have been sought by the program of axiomatic inquiry, which is



IRRATIONAL NUMBERS seemed paradoxical to the Greeks, who could not imagine numbers that were neither integers nor rational fractions but who could nonetheless express such numbers geometrically. In a right triangle with two sides of unit length 3 and 4 respectively, the hypotenuse is 5 units in length. But no rational fraction is equal to $\sqrt{2}$, the length of the hypotenuse of a right triangle that has sides of length 1. In effect, this says that an easily constructed line of quite tangible length is nonetheless "immeasurable."



COMPLEX NUMBERS can be represented and even manipulated in a geometric fashion. On the real, or x, axis each unit is 1 or -1. On the imaginary, or y, axis each unit is i, or $\sqrt{-1}$, or else $-i$. Thus all points on the plane can be given complex numbers of the form $x + yi$. If a line through both the origin and any point on the plane (*as shown in "a"*) is rotated through 90 degrees (*as in "b"*), the result is the multiplication of the original complex number by $\sqrt{-1}$. A second rotation, and multiplication by i, appears in c.

still going on. It has yielded rigorous and aesthetically appealing answers to some of them, and in the process it has brought forth new concepts such as "rings," "fields," "groups" and "lattices," each with its own set of rules of operation and its own characteristic theory.

One of the major accomplishments, achieved in the 1870's, was the establishment of a set of axioms for the real numbers. It is summed up in the statement that the real-number system is a "complete ordered field." Each of these words represents a group of rules that defines the behavior of the numbers.

First of all, the word "field" means a mathematical system in which addition and multiplication can be carried out in a way that satisfies the familiar rules, namely (1) the commutative law of addition: $x + y = y + x$; (2) the associative law of addition: $x + (y + z) = (x + y) + z$; (3) the commutative law of multiplication: $xy = yx$; (4) the associative law of multiplication: $x(yz) = (xy)z$; (5) the distributive law: $x(y + z) = xy + xz$.

Furthermore, a field must contain a zero element, 0, characterized by the property that $x + 0 = x$ for any element x. It contains a unit element, 1, that has the property that $1 \cdot x = x$. For any given element x of a field there is another element $-x$ such that $-x + x = 0$. This is the foundation on which subtraction is built. Another axiomatic property of a field is the cancellation rule of multiplication, that is, if $xy = xz$, then $y = z$ (provided that x is not equal to zero). Finally, for any element x (other than zero) a field must contain an element $1/x$ such that $x(1/x) = 1$. This is the basis for division. Briefly, then, a field is a system (exemplified by the rational numbers) whose elements can be added, subtracted, multiplied and divided under the familiar rules of arithmetic.

Considering now the second word, a field is "ordered" if the sizes of its elements can be compared. The shorthand symbol used to denote this property is the sign $>$, meaning "greater than." This symbol is required to obey its own set of rules, namely (1) the trichotomy law: for any two elements x and y, exactly one of these three relations is true, $x > y$, $x = y$ or $y > x$; (2) the transitivity law: if $x > y$ and $y > z$, then $x > z$; (3) the law of addition: if $x > y$, then $x + z > y + z$; (4) the law of multiplication: if $x > y$ and $z > 0$, then $xz > yz$.

Finally, what do we mean by the word "complete" in describing the system of real numbers as a "complete ordered field"? This has to do with the problem raised by a number such as $\sqrt{2}$. Practically speaking, $\sqrt{2}$ is given by a sequence of rational numbers such as 1, 1.4, 1.41, 1.414... that provide better and better approximations to it. That is to say, $1^2 = 1$, $(1.4)^2 = 1.96$, $(1.41)^2 = 1.9981$, $(1.414)^2 = 1.999396\ldots$. Squaring these numbers yields a sequence of numbers that are getting closer and closer to 2. Notice, however, that the numbers in the original sequence (1, 1.4, 1.41...) are also getting closer and closer to one another. We would like to think of $\sqrt{2}$ as the "limiting value" of such a sequence of approximations. In order to do so we need a precise notion of what is meant by saying that the numbers of a sequence are getting closer and closer to one another, and we need a guarantee that our system of numbers is rich enough to provide us with a limiting number for such a sequence.

Following the path taken by Cantor, we consider a sequence of numbers in our ordered field. We shall say that the numbers of this sequence are getting closer and closer to one another if the difference of any two numbers sufficiently far out in the sequence is as small as we please. This means, for example, that all terms sufficiently far out differ from one another by at the most 1/10. If one wishes to go out still further, they can be made to differ by at most 1/100, and so forth. Such a sequence of numbers is called a "regular sequence." An ordered field is called a "complete" ordered field if, corresponding to any regular sequence of elements, there is an element of the field that the sequence approaches as a limiting value. This is the "law of completeness": the "gaps" between the rational numbers have been completed, or filled up. It is the final axiomatic requirement for the real-number system.

All these rules may seem so elementary that they hardly need stating, let alone laborious analysis. The program of systematizing them, however, has been vastly rewarding. Years of polishing the axioms have reduced them to a form that is of high simplicity. The rules I have just enumerated have been found to be necessary, and sufficient, to do the job of describing and operating the real-number system; throw any one of them away and the system would not work. And, as I have said, the program of axiomatic inquiry has answered some fundamental questions about numbers and produced enormously fruitful new concepts.

The spirit of axiomatic inquiry pervades all modern mathematics; it has even percolated into the teaching of mathematics in high schools. A high school teacher recently said to me: "In the old days the rules of procedure were buried in fine print and largely ignored in the classroom. Today the fine print has been parlayed into the main course. The student is in danger of knowing that $2 + 3 = 3 + 2$ by the commutative law but not knowing that the sum is 5." Of course anything can be overdone. Exclusive attention to axiomatics would be analogous to the preoccupation of a dance group that met every week and discussed choreography but never danced. What is wanted in mathematics, as in anything else, is a sound sense of proportion.

We have been considering how numbers operate; ultimately we must face the more elementary question: What *are* numbers, after all? Nowadays mathematicians are inclined to answer this question too in terms of axiomatics rather than in terms of epistemology or philosophy.

To explain, or better still to create, numbers it seems wise to try the method of synthesis instead of analysis. Suppose we start with primitive, meaningful elements and see if step by step we can build these elements up into something that corresponds to the system of real numbers.

As our primitive elements we can take the positive integers. They are a concrete aspect of the universe, in the form of the number of fingers on the human hand or whatever one chooses to count. As the 19th-century German mathematician Leopold Kronecker put it, the positive integers are the work of God and all the other types of number are the work of man. In the late 19th century Giuseppe Peano of Italy provided a primitive description of the positive integers in terms of five axioms: (1) 1 is a positive integer; (2) every positive integer has a unique positive integer as its successor; (3) no positive integer has 1 as its successor; (4) distinct positive integers have distinct successors; (5) if a statement holds for the positive integer 1, and if, whenever it holds for a positive integer, it also holds for that integer's successor, then the statement holds for all positive integers. (This last axiom is the famous "principle of mathematical induction.")

Now comes the *fiat lux* ("Let there be light") of the whole business. Axiom: There exists a Peano system. This stroke creates the positive integers, because the Peano system, or system of objects that fulfills the five requirements, is essentially equivalent to the set of positive integers. From Peano's five rules all the familiar features of the positive integers can be deduced.

Once we have the positive integers at our disposal to work with and to mold, we can go merrily on our way, as Kronecker suggested, and construct extensions of the number idea. By operations with the positive integers, for example, we can create the negative integers and zero. A convenient way to do this is by operating with *pairs* of positive integers. Think of a general pair denoted (a,b) from which we shall create an integer by the operation $a - b$. When a is greater than b, the subtraction $a - b$ produces a positive integer; when b is greater than a, the resulting $a - b$ integer is negative; when a is equal to b, then $a - b$ produces zero. Thus pairs of positive integers can represent all the integers—positive, negative and zero. It is true that a certain ambiguity arises from the fact that a given integer can be represented by many different pairs; for instance, the pair (6,2) stands for 4, but so do (7,3), (8,4) and a host of other possible combinations. We can reduce the ambiguity to unimportance, however, simply by agreeing to consider all such pairs as being identical.

Using only positive integers, we can write a rule that will determine when one pair is equal to another. The rule is that $(a,b) = (c,d)$ if, and only if, $a + d = b + c$. (Note that the latter equation is a rephrasing of $a - b = c - d$, but it does not involve any negative integers, whereas the subtraction terms may.) It can easily be shown that this rule for deciding the equality of pairs of integers satisfies the three arithmetical laws governing equality, namely (1) the reflexive law: $(a,b) = (a,b)$; (2) the symmetric law: if $(a,b) = (b,c)$, then $(b,c) = (a,b)$; (3) the transitive law: if $(a,b) = (c,d)$ and $(c,d) = (e,f)$, then $(a,b) = (e,f)$.

We can now proceed to introduce

	1	i	j	k
1	1	i	j	k
i	i	−1	k	−j
j	j	−k	−1	i
k	k	j	−i	−1

MULTIPLICATION TABLE for the quaternions, devised by William Rowan Hamilton, demonstrates the noncommutative nature of these imaginary quantities. For example, the row quantity *j*, multiplied by the column quantity *k*, produces *i*, but row *k* times column *j* produces $-i$ instead. Each of the three quantities, multiplied by itself, is equal to −1.

conventions defining the addition and the multiplication of pairs of positive integers, again using only positive terms. For addition we have $(a,b)+(c,d)=(a+c,\ b+d)$. Since (a,b) represents $a-b$ and (c,d) represents $c-d$, the addition here is $(a-b)+(c-d)$. Algebraically this is the same as $(a+c)-(b+d)$, and that is represented by the pair $(a+c,\ b+d)$ on the right side of the equation. Similarly, the multiplication of pairs of positive integers is defined by the formula $(a,b)\cdot(c,d)=(ac+bd,\ ad+bc)$. Here $(a,b)(c,d)$, or $(a-b)(c-d)$, can be expressed algebraically as $(ac+bd)-(ad+bc)$, and this is represented on the right side of the equation by the pair $(ac+bd,\ ad+bc)$.

It can be shown in detail that all the familiar operations with integers (positive, negative and zero), when performed with such pairs of positive integers, will produce the same results.

Having constructed all the integers (as pairs of positive integers), we can go on to create all the other real numbers and even the complex numbers. The rational numbers, or fractions, which are pairs of integers in the ordinary system, can be represented as pairs of pairs of positive integers. For the real numbers, made up of infinite sequences of integers, we must set up infinite sequences of rationals rather than pairs. When we come to the complex numbers, we can again use pairs; indeed, it was for these numbers that the device of number pairs was first employed (by Hamilton). We can think of a complex number, $a+b\sqrt{-1}$, as essentially a pair of real numbers (a,b), with the first number of the pair representing the real element and the second representing the imaginary element of the complex number. Now, pairs will be considered equal only if they contain the same numbers in the same order; that is, $(a,b)=(c,d)$ only if $a=c$ and $b=d$. The rule for addition will be

a

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{pmatrix} = \begin{pmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ a_4+b_4 & a_5+b_5 & a_6+b_6 \\ a_7+b_7 & a_8+b_8 & a_9+b_9 \end{pmatrix}$$

$$\begin{pmatrix} 7 & 0 & 0 \\ -3 & 1 & -6 \\ 4 & 0 & 7 \end{pmatrix} + \begin{pmatrix} -8 & 0 & 1 \\ 4 & 5 & -1 \\ 0 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 7-8 & 0+0 & 0+1 \\ -3+4 & 1+5 & -6-1 \\ 4+0 & 0+3 & 0+7 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 6 & -7 \\ 4 & 3 & 7 \end{pmatrix}$$

b

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} \times \begin{pmatrix} b_1 & b_4 & b_7 \\ b_2 & b_5 & b_8 \\ b_3 & b_6 & b_9 \end{pmatrix} = \begin{pmatrix} a_1b_1+a_2b_2+a_3b_3 & a_1b_4+a_2b_5+a_3b_6 & a_1b_7+a_2b_8+a_3b_9 \\ a_4b_1+a_5b_2+a_6b_3 & a_4b_4+a_5b_5+a_6b_6 & a_4b_7+a_5b_8+a_6b_9 \\ a_7b_1+a_8b_2+a_9b_3 & a_7b_4+a_8b_5+a_9b_6 & a_7b_7+a_8b_8+a_9b_9 \end{pmatrix}$$

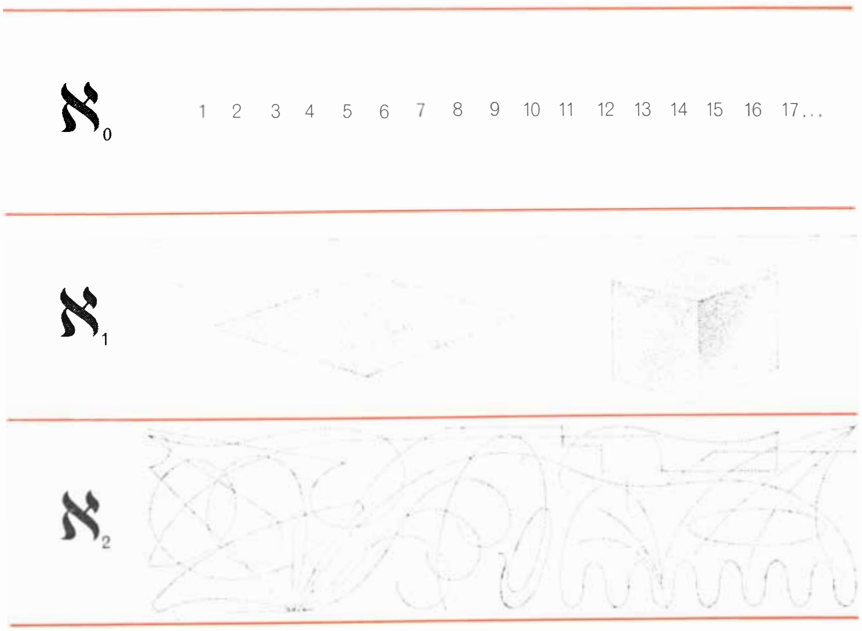
$$\begin{pmatrix} 6 & 0 & -1 \\ 1 & -3 & 2 \\ 8 & 5 & 6 \end{pmatrix} \times \begin{pmatrix} 4 & 2 & 3 \\ 0 & 1 & 6 \\ -5 & -1 & 7 \end{pmatrix} = \begin{pmatrix} 24+0+5 & 12+0+1 & 18+0-7 \\ 4+0-10 & 2-3-2 & 3-18+14 \\ 32+0-30 & 16+5-6 & 24+30+42 \end{pmatrix} = \begin{pmatrix} 29 & 13 & 11 \\ -6 & -3 & -1 \\ 2 & 15 & 96 \end{pmatrix}$$

MATRICES are rectangular arrays of numbers, themselves without numerical value, that nonetheless can be treated as entities and thus can be added, subtracted, multiplied or divided in the proper circumstances. Such arrays offer a particularly convenient method for calculating simultaneous changes in a series of related variables. Addition is possible with any pair of matrices having the same number of columns and rows; row by row, each element in each column of the first matrix is added to the corresponding element in the corresponding column of the next, thus forming a new matrix. (The process is shown schematically at the top of the illustration and then repeated, with numerical values, directly below.) Multiplication is a more complex process, in which the two matrices need not be the same size, although they are in the illustration; a 3×2 matrix could multiply a 2×3 one. Each term in the upper row of the left matrix successively multiplies the corresponding term in the first column of the right matrix; the sum of these three multiplications is the number entered at column 1, row 1 of the product matrix. The upper row of the left matrix is now used in the same way with the second column of the right matrix to find a value for column 2, row 1 of the product matrix, and then multiplies the third column of the right matrix. The entire operation is repeated with each row of the left matrix.

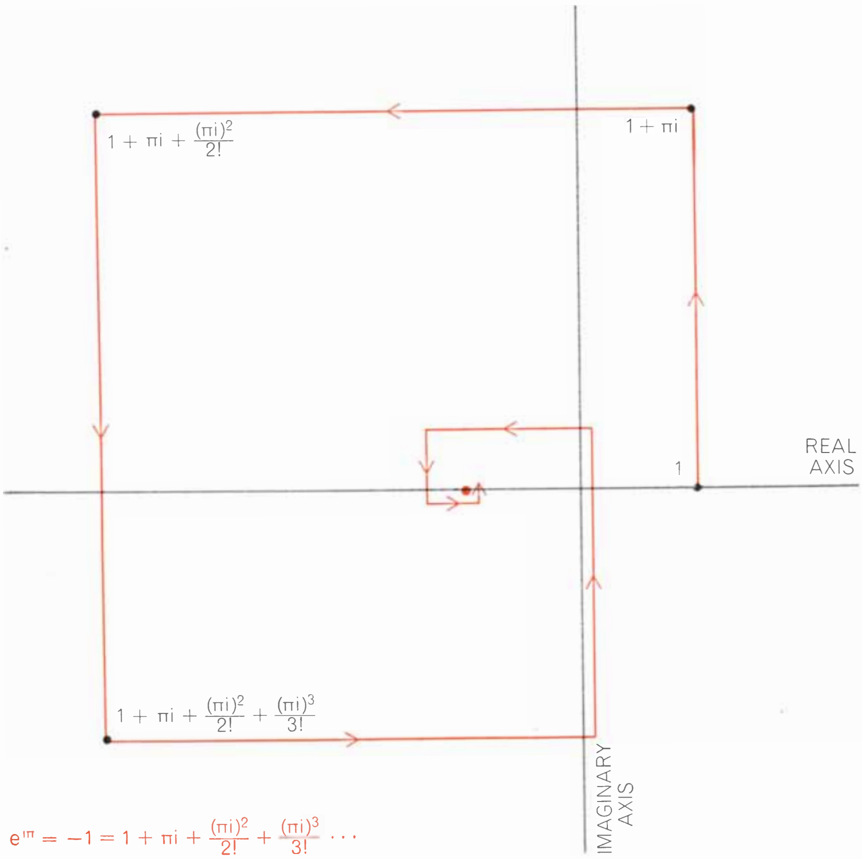
the same as in the case of the real numbers: $(a,b) + (c,d) = (a + c,\ b + d)$. This parallels the "ordinary" outcome of the addition of two complex numbers: $(a + b\sqrt{-1}) + (c + d\sqrt{-1}) = (a + c) + (b + d)\sqrt{-1}$. The multiplication formula for complex numbers, $(a,b) \cdot (c,d) = (ac - bd,\ ad + bc)$, also corresponds to the ordinary multiplication of such numbers: $(a + b\sqrt{-1})(c + d\sqrt{-1}) = (ac - bd) + (ad + bc)\sqrt{-1}$. Pairs of real numbers manipulated according to these rules reproduce all the familiar behavior of the complex numbers. And the mysterious $\sqrt{-1}$, that "amphibian between being and not being," emerges from the sea of axiomatics as the number pair (0,1).

Thus, by four steps of construction and abstraction, we have advanced from the primitive positive integers to the complex numbers. Pairs of positive integers, combined in a certain way, lead to the set of all the integers. Pairs of integers (that is to say now, pairs of pairs of positive integers), combined in a different way, lead to the rational numbers. Infinite sequences of rational numbers lead to real numbers. Finally, pairs of real numbers lead to the complex numbers.

Looking back over the 2,500 years that separate us from Pythagoras, we can make out two streams of thinking about numbers. There is the stream of synthesis, which began with tally marks and went on to build up number concepts of increasing complexity, in much the same way that a complex molecule is built up from atoms. On the other hand, there is a stream of analysis whereby mathematicians have sought to arrive at the essence of numbers by breaking down the complexities to their most primitive elements. Both streams are of enormous importance. Professional mathematicians today tend to play down number as such, favoring the qualitative aspects of their science and emphasizing the logical structure and symbolic potentialities of mathematics. Nevertheless, new ideas about number keep making their way into the mathematics journals, and the modern number theories are just now diffusing rapidly throughout our educational system, even down to the elementary schools. There are programs and committees for teaching advanced number concepts, from set theory to matrices, to students in high school. It seems safe to say that the coming generation will be imbued with an unprecedented interest in the fascinating uses and mysteries of numbers.



TRANSFINITE CARDINALS exist in infinite number. The most familiar, $\aleph_0$, symbolizes the "number," or the cardinality, of the positive integers or of any set that can be put into a one-to-one correspondence with the positive integers. These are the sets that are countable. The cardinality of the real numbers is larger than the cardinality of the positive integers. It is identical with the cardinality of the points on a line, in a plane or in any portion of a space of higher dimension. These noncountable sets are symbolized by $\aleph_1$. The "number" of all possible point sets is a still larger transfinite cardinal and is symbolized by $\aleph_2$.



VIRTUOSITY OF COMPLEX NUMBERS is demonstrated by conversion into geometrical form of the formula that relates e (the natural base of logarithms), π and $\sqrt{-1}$. The equation (*color*) can be expressed as the sum of a series of vectors. When these are added and plotted on a complex plane, they form a spiral that strangles the point equal to -1.

GEOMETRY

For 2,000 years geometry meant Euclidean geometry. Then it was found not only that other geometries described physical space equally well but also that geometry was properly the study of all possible spaces

by Morris Kline

The evolution of mathematics depends on advances in both number and geometry. It cannot be said, however, that these key elements of mathematics have always advanced side by side. Frequently they have competed, and the advance of one has been at the expense of the other. The history of this sometimes strained relation between two disciplines that actually have a common purpose is reminiscent of contrapuntal themes in music.

The first genuine stride of mathematics was taken by geometry. Some primitive mathematics was created by Egyptian and Babylonian carpenters and surveyors in the 4,000 years preceding the Christian era, but it was the classical Greek philosophers who, between 600 B.C. and 300 B.C., gave mathematics its definitive architecture of abstraction and deductive proof, erected the vast structure of Euclidean geometry and dedicated the subject to the understanding of the universe.

Of the several forces that turned the Greeks toward geometry, perhaps the most important was the difficulty Greek scholars had with the concept of the irrational number: a number that is neither a whole number nor a ratio of whole numbers. The difficulty arose in connection with the famous Pythagorean theorem that the length of the hypotenuse of a right triangle is the square root of the sum of the squares of the two sides. In a right triangle with sides of one unit each the hypotenuse must then be $\sqrt{2}$, an irrational number. Such a concept was beyond the Greeks; number to them had always meant whole number or ratio of whole numbers. They resolved the difficulty by banishing it, producing a geometry that affirmed theorems and offered proofs without reference to number. Today this geometry is known as pure geometry or synthetic geometry, the latter an unfortunate term that has only historical justification.

Since the mathematics of the classical Greeks was devoted to deducing truths of nature, it had to be founded on truths. Fortunately there were some seemingly self-evident truths at hand, among them the following: two points determine a line; a straight line extends indefinitely far in either direction; all right angles are equal; equals added to equals yield equals; figures that can be made to coincide are congruent. Some of these axioms make assertions primarily about space itself; others pertain to figures in space.

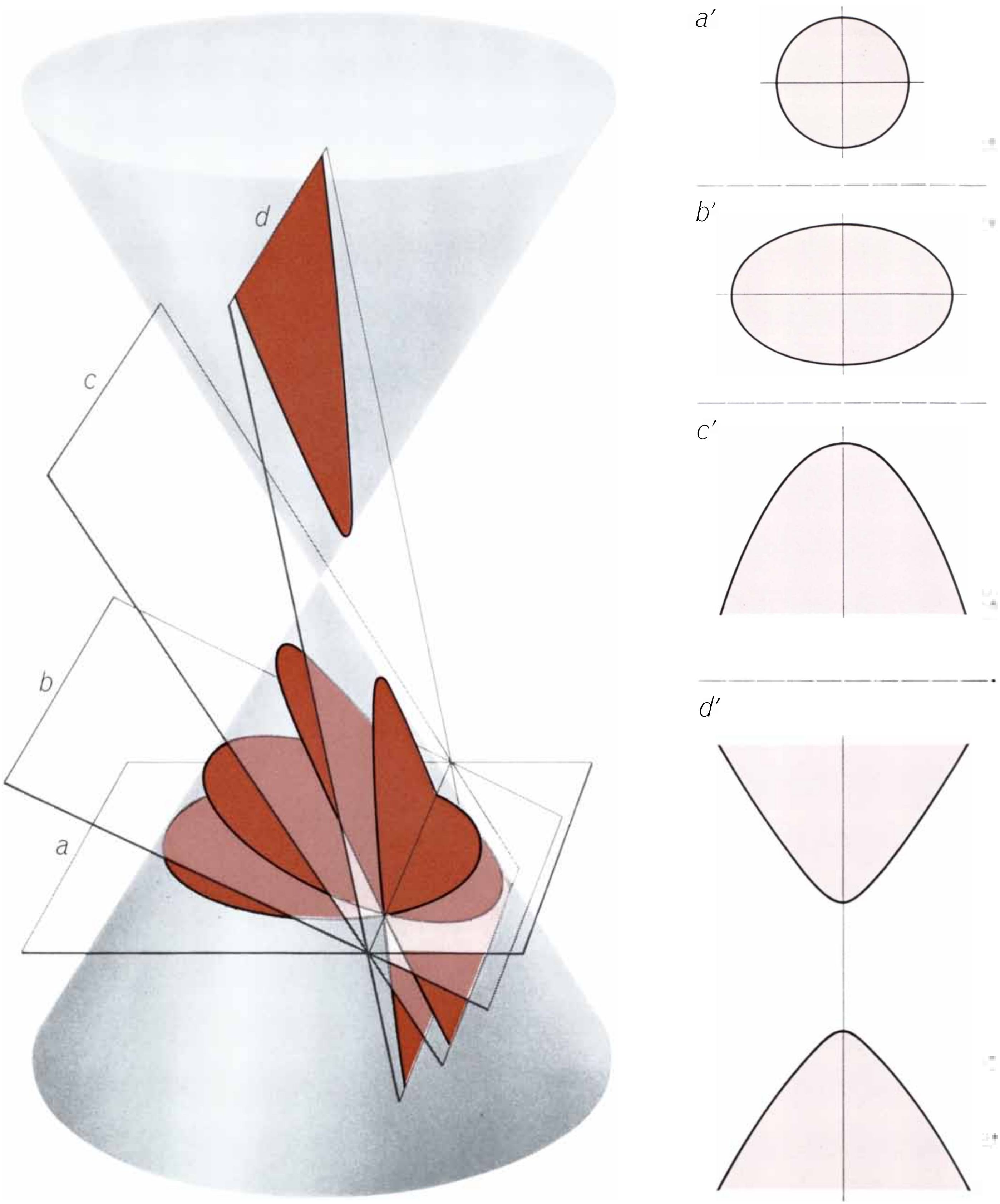
From these axioms Euclid, in his *Elements,* deduced almost 500 theorems. In other works he and his successors, notably Archimedes and Apollonius, deduced many hundreds more. Because the Greeks chose to work purely in geometry, many of the theorems stated results now regarded as algebraic. For example, the solution of second-degree equations in one unknown ($x^2 - 8x + 7 = 0$ is such an equation) was carried out geometrically and the answer given by Euclid was not a number but a line segment. Thus Euclidean geometry embraced the algebra known at that time.

The welter of theorems might suggest that the Greeks drifted from topic to topic. That would be a false impression. The figures they chose were basic: lines and curves in one category and surfaces in another. In the first category are such figures as the triangle and the conic sections: circle, parabola, ellipse and hyperbola. In the second category are such figures as the cube, sphere, paraboloid, ellipsoid and hyperboloid [*see illustration on page 62*]. Then the Greek geometers tackled basic problems concerning those figures. For instance, what must one know about two figures to assert that they are congruent (identical except for position in space), similar (having the same shape if not the same size) or equivalent (having the same area)? Thus congruence, similarity and equivalence are major themes of Euclidean geometry, and the majority of the theorems deal with these questions.

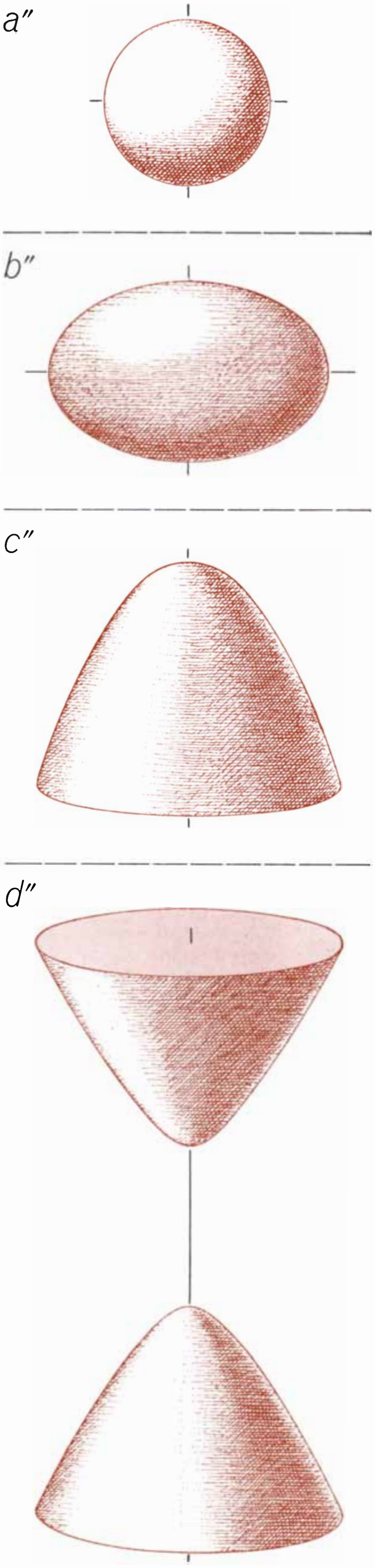
The classical Greek civilization that gave rise to Euclidean geometry was destroyed by Alexander the Great and rebuilt along new lines in Egypt. Alexander moved the center of his empire from Athens to the city he modestly named Alexandria, and he proclaimed the goal of fusing Greek and Near Eastern civilizations. This objective was ably executed by his successors, the Ptolemys, who ruled Egypt from 323 B.C. until the last member of the family, Cleopatra, was seduced by the Romans. Under the influence of the Near Eastern civilizations, notably the Egyptian and the Persian, the culture of the Alexandrian Greek civilization became more engineering-minded and more practically oriented. The mathematicians responded to the new interests.

Applied science and engineering must in large part be quantitative. What the Alexandrians appended to Euclid's geometry in order to obtain quantitative results was number: arithmetic and algebra. The disturbing fact about these

RAPHAEL'S "SPOSALIZIO," or "Marriage of the Virgin," part of which is reproduced on the opposite page, indicates how Renaissance painters solved problems of perspective and so contributed to the evolution of projective geometry. The superimposed white lines show how the artist depicted as converging on a "principal vanishing point" lines that in actuality were horizontal, parallel and receding directly from the viewer.



CONIC SECTIONS provide the basic curves with which geometry deals. By following each series of letters, such as *a*, *a′* and *a″*, one can see at left a plane intersecting a cone to produce a curve, at center the resulting curve and at right the corresponding surface. Thus *a′* is a circle and *a″* a sphere, *b′* an ellipse and *b″* an ellipsoid, *c′* a parabola and *c″* a paraboloid, *d′* a hyperbola and *d″*



a hyperboloid. Definitions and properties of conic sections were worked out by ancient Greek scholars, notably Apollonius.

subjects was that they did not have a logical foundation; the Alexandrians merely took over the empirically based arithmetical knowledge built up by the Egyptians and Babylonians. Because Euclidean geometry offered the security of proof, it continued for centuries to dominate mathematics. Not until late in the 19th century did mathematicians solve the problem of providing an axiomatic basis for arithmetic and algebra.

Actually geometry consists of several geometries. The first break in the direction of a new geometry was made by Renaissance painters who sought to solve the problem of depicting exactly what the eyes see. Because real scenes are three-dimensional, whereas a painting is flat, it would appear to be impossible to paint realistically. The painters solved their problem by recognizing a fundamental fact about vision. Suppose a man, using one eye, looks through a window at some real scene. He sees the scene because light rays from various points in it travel to his eye. This collection of light rays is called a projection. Since the rays pass through the window, it is possible to mark a point on the window where each light ray pierces it. This collection of points is called a section. What the painters discovered is that the section creates the same impression on the eye as the scene itself does. This is physically understandable [*see top illustration on next page*]. Whether the light rays emanate from particles in the real scene or from points on the window, the same light rays reach the eye. Hence the canvas could contain what appears on the window. Even though this is a one-eye scheme and sight involves two eyes, the painters compensated for the restriction by using diminution of light intensity with distance and by using shadows. How well they succeeded in solving the problems of perspective can be judged by the painting reproduced on page 61.

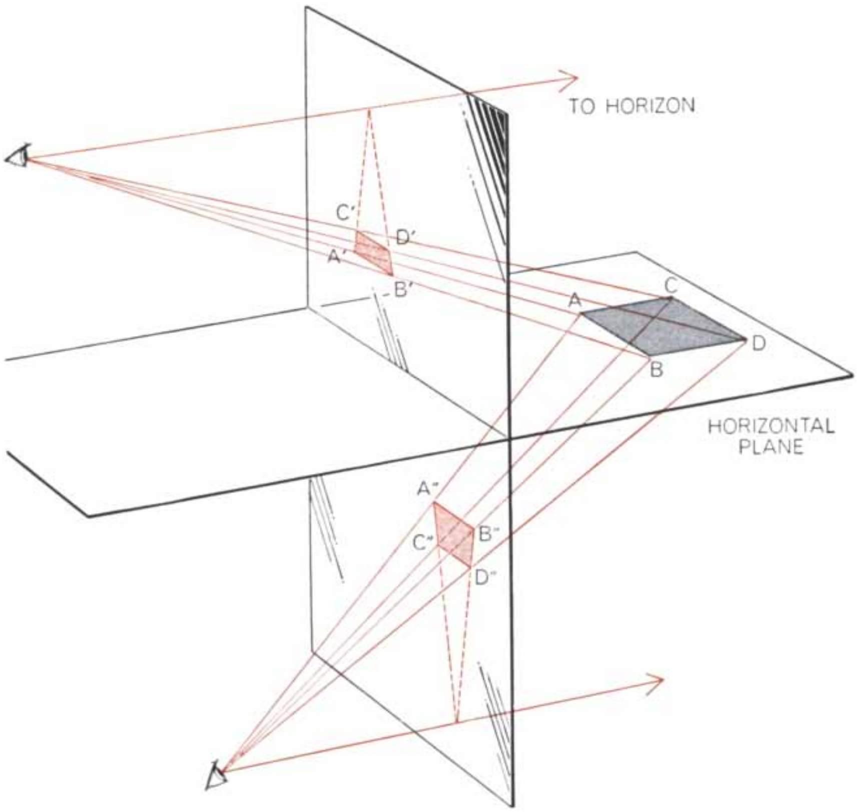
The use of projection and section raised a basic geometrical question, first voiced by the painters and later taken up by mathematicians. What geometrical properties do an original figure and its section have in common that enable them to create the same impression on the eye? The answer to this question led to new concepts and theorems that ultimately constituted a new branch of geometry called projective geometry [see "Projective Geometry," by Morris Kline; SCIENTIFIC AMERICAN, January, 1955]. Some of the concepts and theorems are as follows. It is apparent from the top illustration on the next page that the section of the projection of a line is a line and that, if two lines intersect, then a section of the projection of these two intersecting lines will also be two intersecting lines, although the angle between the two lines of the section will generally not be the same as the angle between the two lines in the original figure. It follows that a triangle will give rise to a triangular section and a quadrilateral will give rise to a quadrilateral section.

A more significant example of the properties common to a figure and a section was furnished in the 17th century by the self-educated French architect and engineer Gérard Desargues. In what is now known as Desargues's theorem he showed that for any triangle and any section of any projection of that triangle any pair of corresponding sides will meet in a point and the three points of intersection of the three pairs of corresponding sides lie on one straight line [*see bottom illustration on next page*]. The significance of this and other theorems of projective geometry is that this geometry no longer discusses congruence, similarity, equivalence and other concepts of Euclidean geometry but instead deals with collinearity (points that lie on a line), concurrency (lines that go through a point) and other notions stemming from projection and section.

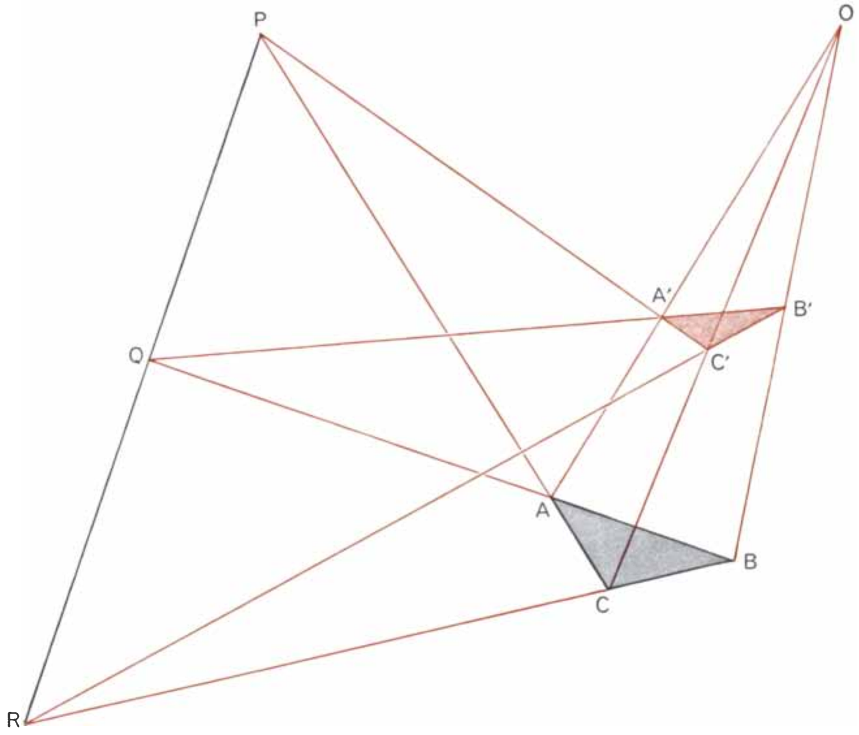
Projective geometry flourished rather briefly and then was pushed aside temporarily by a rival geometry that appeared on the scene. The rival, which embodied an algebraic approach to geometry, is now called analytic geometry or coordinate geometry. It was motivated by a series of events and discoveries that in the 16th and 17th centuries launched the scientific age in western Europe and brought to the fore the problem of deriving and using the properties of curves and surfaces.

For one thing, the creation by Nicolaus Copernicus and Johannes Kepler of the heliocentric theory of planetary motion made manifest the need for effective methods of working with the conic sections; these curves are the paths of the celestial bodies in such a system. Moreover, by invalidating classical Greek mechanics, which presupposed a stationary earth, the heliocentric theory necessitated a completely new science of motion and therefore the study of curves along which objects move.

Several other forces pushed geometry



PROJECTION AND SECTION were concepts that arose from the work of artists and helped lead to projective geometry. Projections of a square, such as *ABCD*, to two observers form sections (*color*) on an intersecting plane. In a drawing the square must be represented as a section in order to appear realistic to an observer looking at the drawing.



DESARGUES'S THEOREM illustrates the concern of projective geometry with properties common to a figure and its sections. The theorem states that any pair of corresponding sides of a triangle (*ABC*) and a section (*color*) will meet in a point—as, for example, the sides *BC* and *B′C′* meet in point *R*—and that the three points *P*, *Q* and *R* will lie on a line.

in the same direction. The gradually increasing use of gunpowder raised problems of projectile paths. The discovery of the telescope and the microscope motivated the study of lenses. Geographical exploration called for maps and in particular for the correlation of paths on the globe with paths on flat maps. All these problems not only increased the need for knowledge of properties of familiar curves but also introduced new curves. As René Descartes and Pierre de Fermat realized, the Euclidean synthetic methods were too limited to deal with these problems.

Descartes and Fermat, both major contributors to the fast-growing discipline of algebra, saw the potentialities in that subject for supplying methodology to geometry. The analytic geometry they developed replaced curves by equations through the device of a coordinate system. Such a system locates points in a plane or in space by numbers. In a plane the system uses two numbers, an abscissa and an ordinate [*see illustration on opposite page*]. The abscissa expresses the distance of a point from a fixed vertical line, called the Y axis; the ordinate expresses the distance of the point from a fixed horizontal line, called the X axis. Distances to the right of the Y axis or above the X axis are positive; distances in the opposite directions are negative.

How does this device enable one to represent curves algebraically? Consider a circle with a radius of five units. A circle, like any other curve, is just a particular collection of points. And if the circle is placed on a coordinate system, then each point on the circle has a pair of coordinates. Since the circle is a particular collection of points, the coordinates of these points are special in some way. The specialized nature is expressed by the equation $x^2 + y^2 = 5^2$. What this equation states is that if one takes the abscissa of any point on the curve and substitutes that for x, and if one takes the ordinate of that same point and substitutes it for y, then the number obtained for x^2+y^2 will be 25. One says that the coordinates of any point on the curve satisfy the equation. Moreover, the coordinates of only those points that do lie on the curve satisfy the equation. In the case of surfaces an equation in three coordinates serves. For example, the equation of a sphere with a five-unit radius is $x^2 + y^2 + z^2 = 25$.

Thus under the Descartes-Fermat scheme points became pairs of numbers, and curves became collections of pairs of numbers subsumed in equations. The properties of curves could be deduced by algebraic processes applied to the equations. With this development the relation between number and geometry had come full circle. The classical Greeks had buried algebra in geometry, but now geometry was eclipsed by algebra. As the mathematicians put it, geometry was arithmetized.

Descartes and Fermat were not entirely correct in expecting that algebraic techniques would supply the effective methodology for working with curves. For instance, those techniques could not cope with slope and curvature, which are fundamental properties of curves. Slope is the rate at which a curve rises or falls per horizontal unit; curvature is the rate at which the direction of the curve changes per unit along the curve. Both rates vary from point to point along all curves except the straight line and the circle. To calculate rates of change that vary from point to point the purely algebraic techniques of Descartes and Fermat are not adequate; the calculus, particularly the differential calculus, must be employed. Indeed, the distinguishing feature of the calculus is its power to yield such rates.

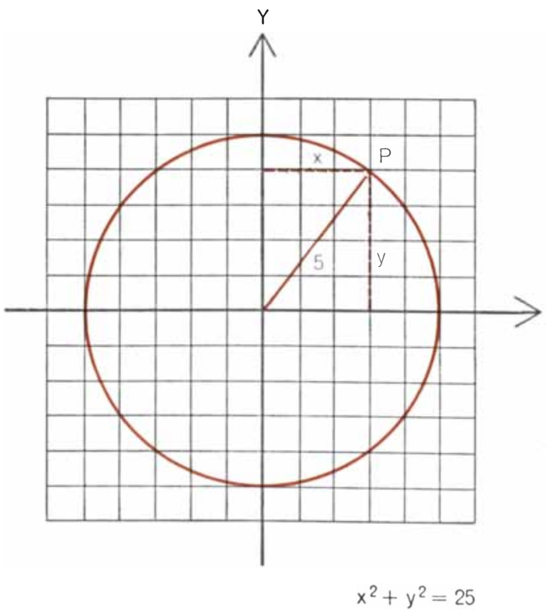
With the aid of the differential calculus the study of curves and surfaces was expedited so much that a new term, differential geometry, was introduced to designate this study. Differential geometry considers a variety of problems beyond the calculation of slope and curvature. It considers in particular the all-important problem of geodesics, or the shortest distance between two points on a surface. Given a surface such as the surface of the earth, what curve joining two given points P and Q on the surface is the shortest distance from P to Q along the surface? If one takes the surface of the earth to be a sphere, the answer is simple. The geodesics are arcs of great circles. (A great circle cuts the sphere in half; the Equator is a great circle but a circle of latitude is not.) If one more accurately takes the surface of the earth to be an ellipsoid, however, the geodesics are more complicated curves and depend on which points P and Q one chooses. The concerns of differential geometry include the curvature of surfaces, map making and surfaces of least area bounded by curves in space, the last of which are so handsomely realized by soap films [*see bottom illustration on page 67*].

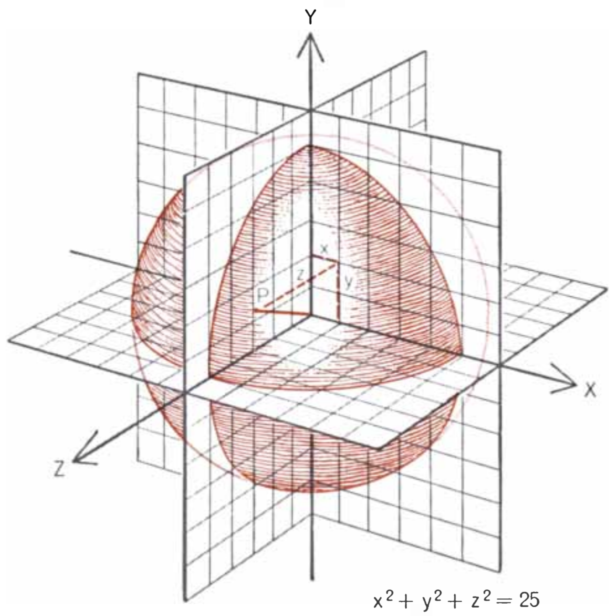
From the standpoint of pure geometry the methodologies of analytic geometry and differential geometry were far too successful. Although these subjects treated geometry, the representations of curves were equations and the methods of proof were algebraic or analytic (that is, they involved the use of the calculus). The beautiful geometrical reasoning was abandoned and geometry was submerged in a sea of formulas. The spirit of geometry was banished.

For 150 years the pure geometers remained in the shadows. In the 19th century, however, they found the courage and the vitality to reassert themselves. The revival of geometry was launched by Gaspard Monge (1746–1818), a leading French mathematician and adviser to Napoleon. Monge thought the analysts had sold geometry short and had even handicapped themselves by failing to interpret their analysis geometrically and to use geometrical pictures to help them think. Monge was such an inspiring teacher that he gathered about him a number of very bright pupils, among them L. N. M. Carnot (1753–1823), Charles J. Brianchon (1785–1864) and Jean Victor Poncelet (1788–1867). These men, imbued by Monge with a fervor for geometry, went beyond the intent of their master and sought to show that geometric methods could accomplish as much and more than the algebraic and analytic methods. To defeat Descartes or, as Carnot put it, "to free geometry from the hieroglyphics of analysis," became the goal.

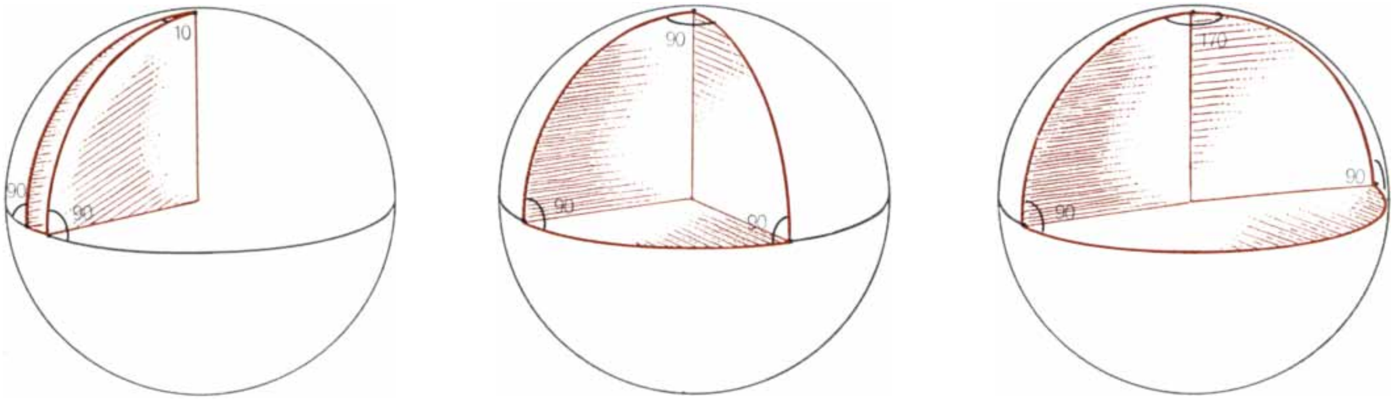
The geometers, led by Poncelet, turned back to projective geometry, which had been so ruthlessly abandoned in the 17th century. Poncelet, serving as an officer in Napoleon's army, was captured by the Russians and spent the year 1813–1814 in a Russian prison. There he reconstructed without the aid of any books all he had learned from Monge; he then proceeded to create new results in projective geometry.

Projective geometry was actively pursued throughout the 19th century. Curiously an algebraic method, essentially an extension of the method of coordinate geometry, was developed to prove





CARTESIAN COORDINATE SYSTEM made it possible to express any shape as an equation. For the circle at left, with a radius of five units, the equation is $x^2 + y^2 = 25$. Any values of x and y that produced 25 in the equation would represent a point on this circle; for the point P, $x = 3$ and $y = 4$. At right is a visualization of the sphere represented by the equation $x^2 + y^2 + z^2 = 25$.



SPHERICAL TRIANGLES can have angles that sum to more than 180 degrees. On the sphere at left the triangle has angles summing to 190 degrees. On the succeeding spheres the angles of the triangles sum respectively to 270 degrees, 350 degrees and 510 de-

its theorems, and to this extent the interests of the pure geometers who launched the revival were subverted. But projective geometry was again put in the shade by another development as dramatic and as weighty as the creation of mathematics by the classical Greeks: the creation of non-Euclidean geometry.

Throughout the long reign of Euclidean geometry many mathematicians were troubled by a slight blemish that seemed to mar the collection of axioms. Apropos of parallel lines, by which is meant two lines in the same plane that do not contain any points in common, Euclid formulated an axiom that reads as follows: If the straight line *n* cuts the lines *l* and *m* so as to make corresponding angles with each line that total less than 180 degrees, then *l* and *m* will meet on that side of the line *n* on which the angles lie. This axiom is essential to the derivation of the most important theorems, among them the theorem that the sum of the angles of a triangle is 180 degrees. The axiom is a bit involved, and there are reasons to believe Euclid himself was not too happy about it. Neither he nor any of the later mathematicians up to about 1800 really doubted the truth of the statement; that is, they had no doubt that it was a correct idealization of the behavior of actual, or physical, lines. What bothered Euclid and his successors was that the axiom was not quite so self-evident as, say, the axiom that any two right angles are equal.

From Greek times on mathematicians sought to replace the axiom on parallels by an equivalent one: an axiom that, together with the other nine axioms of Euclid, would make it possible to deduce the same body of theorems Euclid deduced. Many equivalent axioms were proposed. One of these, which was suggested by the mathematician John Playfair (1748–1819) and is the one usually taught in high schools, states that given a line *l* and a point *P* not on *l*, there is only one line *m* in the plane of *P* and *l* that passes through *P* and does not meet *l*.

Playfair's axiom is not only equivalent to Euclid's axiom but it is also simpler and appears to be intuitively convincing; that is, it does seem to state an unquestionable or self-evident property of lines in physical space. Later mathematicians, however, were not satisfied with Playfair's axiom or any of the other proposed equivalents of Euclid's axiom. The reason they were not satisfied was that every proposed substitute directly or indirectly involved an assertion about what happens far out in space. Thus Playfair's axiom asserts that *l* and *m* will not meet, no matter how far out these lines are extended. As a matter of fact, Euclid's axiom is superior in this respect because all it asserts is a condition under which lines will meet at some finite distance.

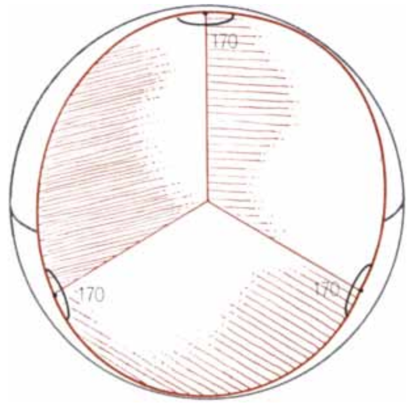
What is objectionable about axioms that assert what happens far out in space? The answer is that they transcend experience. The axioms of Euclidean geometry were supposed to be unquestionable truths about the real world. How can one be sure that two straight lines will extend indefinitely far out into physical space without ever being forced to meet? The problem the mathematicians faced was that Euclid's parallel axiom was not quite self-evident, and that the equivalent axioms, which were seemingly more self-evident, proved on closer examination to be somewhat suspect also.

The problem of the parallel axiom or, as the French mathematician Jean Le Rond d'Alembert put it, "the scandal of geometry," engaged the mathematicians of every period from Greek times up to 1800. The history of these investigations would be worth noting if for no other reason than to see how persistent and critical mathematicians can be. It is necessary here to forgo the history and jump to the results. The truth that destroyed truth was seen clearly by the greatest of all 19th-century mathematicians, Karl Friedrich Gauss (1777–1855). His first point was somewhat technical but essential, namely, that the parallel axiom is independent of the other nine axioms; that is, it is logically possible to choose a contradictory axiom and use it in conjunction with the other nine Euclidean axioms to deduce theorems of a new geometry. Thus one might assume that given a line *l* and a point *P* not on *l*, there is an infinite number of lines through *P* and in the plane of *P* and *l* that do not meet *l*. Gauss adopted this very axiom and from it and the other nine axioms deduced a number of theorems. Gauss called his new geometry non-Euclidean geometry.

As might be expected, many theorems of the new geometry contradict theorems of Euclidean geometry. The sum of the angles of a triangle in this geometry is always less than 180 degrees. Moreover, the sum varies with the size of the triangle; the closer the area of the triangle is to zero, the closer the angle sum is to 180 degrees.

The existence of a logical alternative to Euclidean geometry was in itself a startling fact. Geometry up to this time had been essentially Euclidean geometry; analytic and differential geometry were merely alternative technical methodologies, and although projective geometry dealt with new concepts and new themes, they were entirely in accord with Euclidean geometry. Non-Euclidean geometry was in conflict with Euclidean geometry.

Gauss's second conclusion was even more disturbing. It was that non-Eu-



grees. Such triangles typify concepts of Bernhard Riemann's non-Euclidean geometry.

clidean geometry could be used to represent physical space just as well as Euclidean geometry does. This assertion seems at first to be downright nonsense. If the sum of the angles of a triangle is 180 degrees, how could it also be less than 180 degrees? The answer to this seeming impossibility is that the non-Euclidean geometry calls for an angle sum arbitrarily close to 180 degrees when the size of the triangle is small enough. The triangles man usually deals with are small; therefore the angle sums of these triangles might be so close to 180 degrees that measurement of the sum, in view of the inevitable errors of measurement, would not exclude either possibility.

The implications of non-Euclidean geometry are drastic. If both Euclidean and non-Euclidean geometry can represent physical space equally well, which is the truth about space and figures in space? One cannot say. In fact, the choice might not be limited to just these two. This doleful possibility was soon to be realized. The fact gradually forced on the mathematicians is that geometry is not the truth about physical space but the study of possible spaces. Several of these mathematically constructed spaces, differing sharply from one another, could fit physical space equally well as far as experience could decide.

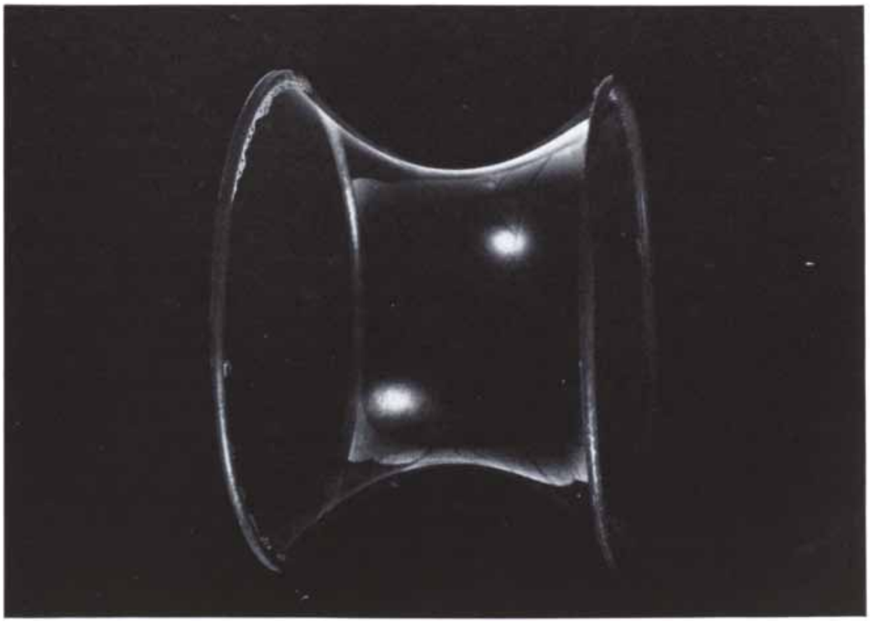
The concept of geometry had then to be revised, but the same was true for the concept of mathematics itself. Since for more than 2,000 years mathematics had been the bastion of truth, non-Euclidean geometry, the triumph of reason, proved to be an intellectual disaster. This new geometry drove home the idea that mathematics, for all its usefulness in organizing thought and advancing the works of man, does not offer truths but is a man-made fable having the semblance of fact.

The new vista opening up in geometry was widened immeasurably by the work of Georg Friedrich Bernhard Riemann (1826–1866). Riemann was one of Gauss's students and undoubtedly acquired from him an interest in the study of the physical world. Riemann's first observation in the field of geometry was that the mathematicians had been deceived into believing the Euclidean parallel axiom was necessarily true. Perhaps they were equally deceived in accepting one or more of the other axioms of Euclid. Riemann fastened at once on the axiom that a straight line is infinite. Experience, he pointed out, does not assure us of the infinitude of the physical straight line. Experience tells us only that in following a straight line we do not come to an end. But neither would one come to an end if one followed the Equator of the earth. In other words, experience tells us only that the straight line is endless or unbounded. If we change the relevant axiom of Euclid accordingly, and if we assume that there are *no* parallel lines, we have another set of axioms from which we can deduce still another non-Euclidean geometry.

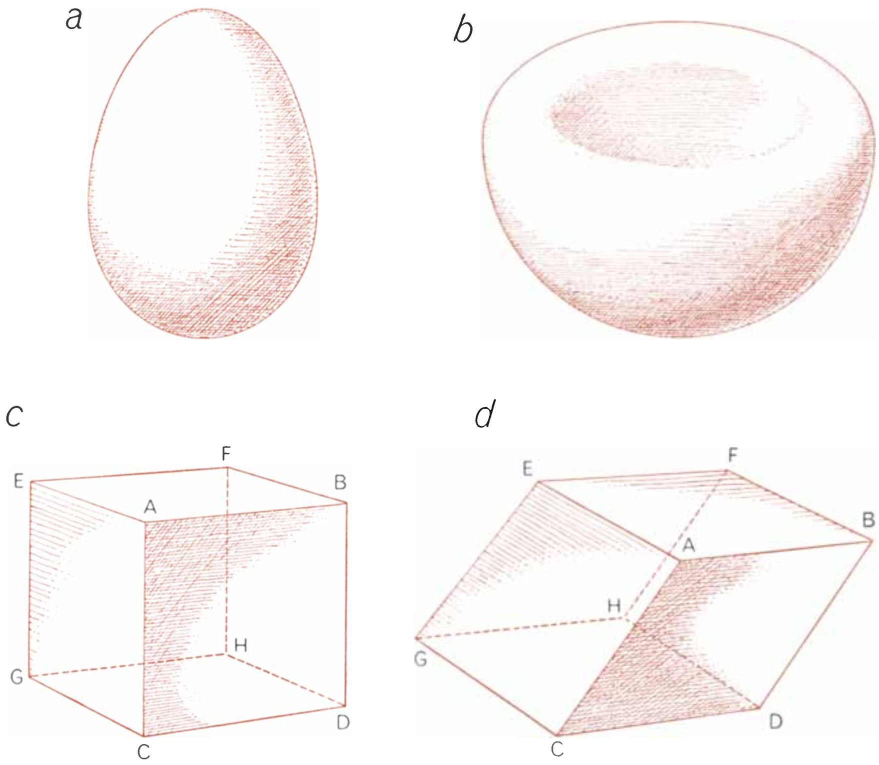
In a paper of 1854 entitled "On the Hypotheses Which Underlie Geometry" Riemann launched an even deeper investigation of possible spaces, utilizing only the surest facts about physical space. He constructed a new branch of geometry, now known as Riemannian geometry, that opened up the variety of mathematical spaces a thousandfold [see "The Curvature of Space," by P. Le Corbeiller; SCIENTIFIC AMERICAN, November, 1954].

To appreciate Riemannian geometry one must first perceive that what is chosen as the distance between two points determines the geometry that results. This can be readily seen. Consider three points on the surface of the earth. One can take as the distance between any two the length of the ordinary straight-line segment that joins them through the earth. In this case one obtains a triangle that has all the properties of a Euclidean triangle. In particular, the sum of the angles of this triangle is 180 degrees. One could, however, take as the distance between any two points the distance along the surface of the earth, meaning the distance along the great circle through these points. In this case the three points determine what is called a spherical triangle. Such triangles possess quite different properties. For example, the sum of the angles in them can be any number between 180 degrees and 540 degrees [*see illustration at the top of these two pages*]. This is a fact of spherical geometry.

What Riemann had in mind was a geometry for changing configurations. Suppose one were to try to design a geometry that would fit the surface of a mountainous region. In some places the surface might be flat, in others there might be conical hills and in still others hemispherical hills. The character of the surface changes from place to place,



SOAP FILMS, which always assume a shape with the least possible area, illustrate a concern of differential geometry: surfaces of least area bounded by curves in space. Differential geometry is also applicable to problems of map making and curvature of surfaces.



TOPOLOGICAL DEFORMATIONS of familiar shapes are portrayed. A sphere can be deformed into an egg shape (*a*), a squashed-ball shape (*b*), a cube (*c*) and a deformation of the cube (*d*). Each deformation is topologically equivalent to the others and to the sphere.

and so the distance formula that determines the geometry must change from place to place and possibly even from point to point. Riemann proposed, in other words, nonhomogeneous spaces—spaces whose characteristics vary from point to point or spaces with varying curvature.

Riemann died at the age of 40 and was therefore able to do little more than sketch the broad outline of his conception of space. The further development of Riemannian geometry became the task of many men and is still under way. Early in this century the Italian mathematicians Gregorio Ricci and Tullio Levi-Civita made significant contributions. Ricci introduced the tensor calculus, a formalism that enables one to express geometric relations independently of the coordinate system. Levi-Civita brought a concept of parallelism to Riemannian geometry: it provided a way of expressing the Euclidean notion of parallelism for more general spaces.

The creation of the general theory of relativity by Albert Einstein not only stimulated further work in Riemannian geometry but also suggested the problem of unifying gravitation and electromagnetism in one mathematical framework. Toward this end Hermann Weyl in 1918 introduced what he called affinely connected spaces, a concept that uses Levi-Civita's notion of parallelism rather than the notion of distance to relate the points of a space to one another. An expression of distance even more generalized than Riemann's produced the spaces called Finsler spaces.

Riemann was also the founder of topology, another branch of geometry in which research is most active today. During the 1850's he was working with what are now called functions of a complex variable, and he introduced a class of surfaces, called Riemann surfaces, to represent such functions. The properties of the functions proved to be intimately connected with the geometric properties of the surfaces. For any given function, however, the precise shape of the surface was not critical, and so he found it desirable to classify surfaces in accordance with a new principle.

Given two similar figures, for example a large and a small triangle having the same shape, either can be regarded as a deformation, or transformation, of the other, the change being a uniform expansion of the smaller to obtain the larger, or a uniform contraction of the larger to obtain the smaller. Under projection and section the deformation of one figure into another is more radical. Yet even in these deformations a quadrilateral, say, remains a quadrilateral. It is possible to make still more radical deformations. For instance, a circle can be deformed by being bent into an ellipse or into an even more complicated shape, and a sphere can be stretched to assume the shape of an egg. For Riemann's purposes the circle could be replaced by the ellipse and the sphere by the egg shape. On the other hand, a circle, a figure eight and a trefoil were not interchangeable curves, and the sphere, the doughnut-shaped torus and the pretzel-shaped double torus were not interchangeable surfaces.

Hence Riemann was led to consider deformations that permit stretching, bending, contracting and even twisting. Figures that can be obtained from one another by such deformations are said to be homeomorphic, or topologically equivalent. If, however, one tears a figure or contracts it in such a way as to make points coalesce, the new figure is not topologically equivalent to the old one. Thus one can pinch a circle top and bottom and form a figure eight, but the latter is not topologically the same as the original. It is also possible to describe topologically equivalent figures by imagining them to be made of rubber. Then any figure that can be obtained by stretching, bending or contracting but not tearing the rubber would be topologically equivalent to the initial one.

The major problem of topology is to know when two figures are topologically equivalent. This may be difficult to see by looking at the figures, particularly since topology considers three-dimensional and even higher-dimensional figures. For this reason and others one seeks to characterize equivalent figures by some definitive properties so that if two figures possess these properties, they must be topologically equivalent, just as the congruence of two triangles is guaranteed if two sides and the included angle of one triangle are equal to the respective parts of the other. For example, if one draws any closed curve on the surface of a sphere or on an ellipsoid, the curve bounds a region on the surface. This is not true on the torus [*see illustration on opposite page*]. The sphere and the torus are therefore not topologically equivalent. It is possible to characterize closed surfaces in terms of curves that do or do not bound on the surface, but this criterion will not suffice for more complicated surfaces or for higher-dimensional figures.

Although many basic problems of topology remain unsolved, mathematicians make progress where they can, and in the past 10 years they have turned to the branch called differential topology. In this endeavor they combine the methods of topology and of differential geometry in the hope that two tools will be better than one.

Another enormously active field today is algebraic geometry. Two hundred years ago this subject was an extension of coordinate geometry and was devoted to the study of curves that are more complicated than the conic sections and are represented by equations of degree higher than the second. Since the latter part of the 19th century, however, the proper domain of algebraic geometry has been regarded as the study of the properties of curves, surfaces and higher-dimensional structures defined by algebraic equations and invariant under rational transformations. Such transformations distort a figure more than projective transformations and less than topological transformations do.

Mathematicians, yielding to their propensity to complicate and to algebraicize, have allowed the coordinates in the equations of algebraic geometry to take on complex values and even values in algebraic fields [see "Number," page 50, and "Algebra," page 70]. Consequently even the simple equation $x^2 + y^2 = 25$, which when x and y have real values represents the circle discussed previously, can represent a complicated Riemann surface or a structure so unconventional that it can hardly be imagined. The geometry suffers, but the algebra flourishes.

This discussion of geometry as the study of the properties of space and of figures in space may have exhibited the growth, variety and vitality of geometry and the interconnections of the branches with each other and with other divisions of mathematics, but it does not present the full nature of modern geometry. It is often said that algebra is a language. So is geometry.

Today mathematicians pursue the subject of abstract spaces, and one might infer from the term that the pursuit involves some highly idealized, esoteric spaces. This is true, but the major use of the theory of abstract spaces—indeed, historically the motivation for its study—is to expedite the use of classes of functions in analysis. The "points" of an abstract space are usually functions, and the distance between two points is some significant measure of a difference between two functions. Thus one might be interested in studying functions such as x^2, $3x^2$ and $x^3 - 2x$ and be interested in the values of these functions as x varies from 0 to 1. One could define the distance between any two of these functions as the largest numerical difference between the two for all values of x between 0 and 1. Such function spaces prove to be infinite-dimensional. The Hilbert spaces and Banach spaces about which one hears much today are function spaces. On the mathematical side these are important in the subject known as functional analysis, which is now the chief tool in quantum mechanics.

Why talk about spaces when one is really dealing with functions? It is because the geometrical mode of thinking is helpful and even suggestive of theorems about functions. What may be complicated and obscure when formulated analytically may in the geometrical interpretation be intuitively obvious. The study of abstract spaces is, surprisingly, part of topology because the properties of these structures that are important, whether the structures are regarded as actual spaces or as collections of functions, are preserved, or invariant, under topological transformations.

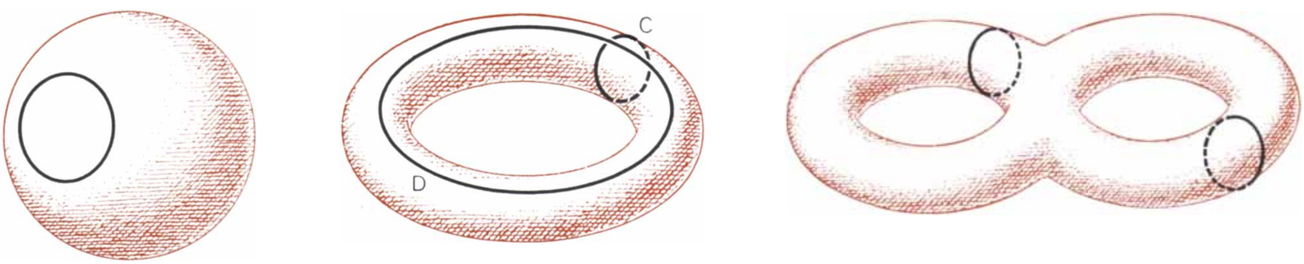
The subject of abstract spaces clearly exhibits the abstractness of modern mathematics. Geometry supplies models not only of physical space but also of any structure whose concepts and properties fit the geometric framework.

In still another vital respect geometry proves to be far more than the receptacle for matter. The present century is witnessing the realization of an assertion by Descartes that physics could be geometrized. In the theory of relativity, one of the two most notable scientific advances of this century (quantum theory is the other), the gravitational effect of gross matter has been reduced to geometry. Just as the geometry of a mountainous region requires a distance formula that varies from place to place to represent the varying shape of the land, so Einstein's geometry has a variable distance formula to represent the different masses in space. Matter determines the geometry, and the geometry as a result accounts for phenomena previously ascribed to gravitation.

Geometry has ingested part of reality and may have to ingest all of it. Today in quantum mechanics physicists are striving to resolve the seemingly contradictory wave and particle properties of subatomic matter, and they may have to generate both from quanta of space. Perhaps matter itself will also dissolve into pure space.

If one assesses today the competition between number and geometry, one must admit that insofar as methodology of proof is concerned, geometry has largely given way to algebra and analysis. The geometric treatment of complicated structures and of course of higher-dimensional spaces can, as Descartes complained of Euclidean geometry, "exercise the understanding only on condition of greatly fatiguing the imagination." Moreover, the quantitative needs of science can be met only by ultimate recourse to number.

Geometry, however, supplies sustenance and meaning to bare formulas. Geometry remains the major source of rich and fruitful intuitions, which in turn supply creative power to mathematics. Most mathematicians think in terms of geometric schemes, even though they leave no trace of that scaffolding when they present the complicated analytical structures. One can still believe Plato's statement that "geometry draws the soul toward truth."



TOPOLOGICAL EQUIVALENCE of surfaces can be determined by drawing closed curves on the figures. If each curve bounds an area on a surface, the surface is topologically equivalent to a sphere. The type of curve drawn on the sphere at left does not bound an area on the torus at center or on the double torus at right; thus the latter figures are not topologically equivalent to a sphere.

ALGEBRA

Elementary algebra deals exclusively with the general properties of numbers. Higher algebra can deal effectively with anything and on occasion is pursued without reference to anything in particular

by W. W. Sawyer

The history of the branch of mathematics known as algebra can be divided into two main epochs. The first lasted from the time of the ancient Egyptian and Babylonian civilizations until about A.D. 1800; the second extends from 1800 to the present. During the earlier epoch men thought about mathematics exclusively in terms of the things with which it dealt: geometry was about shapes, arithmetic was about numbers and algebra was about the relations and properties of numbers in general (as expressed by arbitrary symbols, usually letters of the alphabet). Trigonometry did not fit very neatly into this scheme, since it applied both arithmetic and algebra to geometric problems; nor did analytic geometry, which made geometry a branch of algebra. Nonetheless, the role of algebra appeared more or less clear throughout the epoch: x *always* stood for a number.

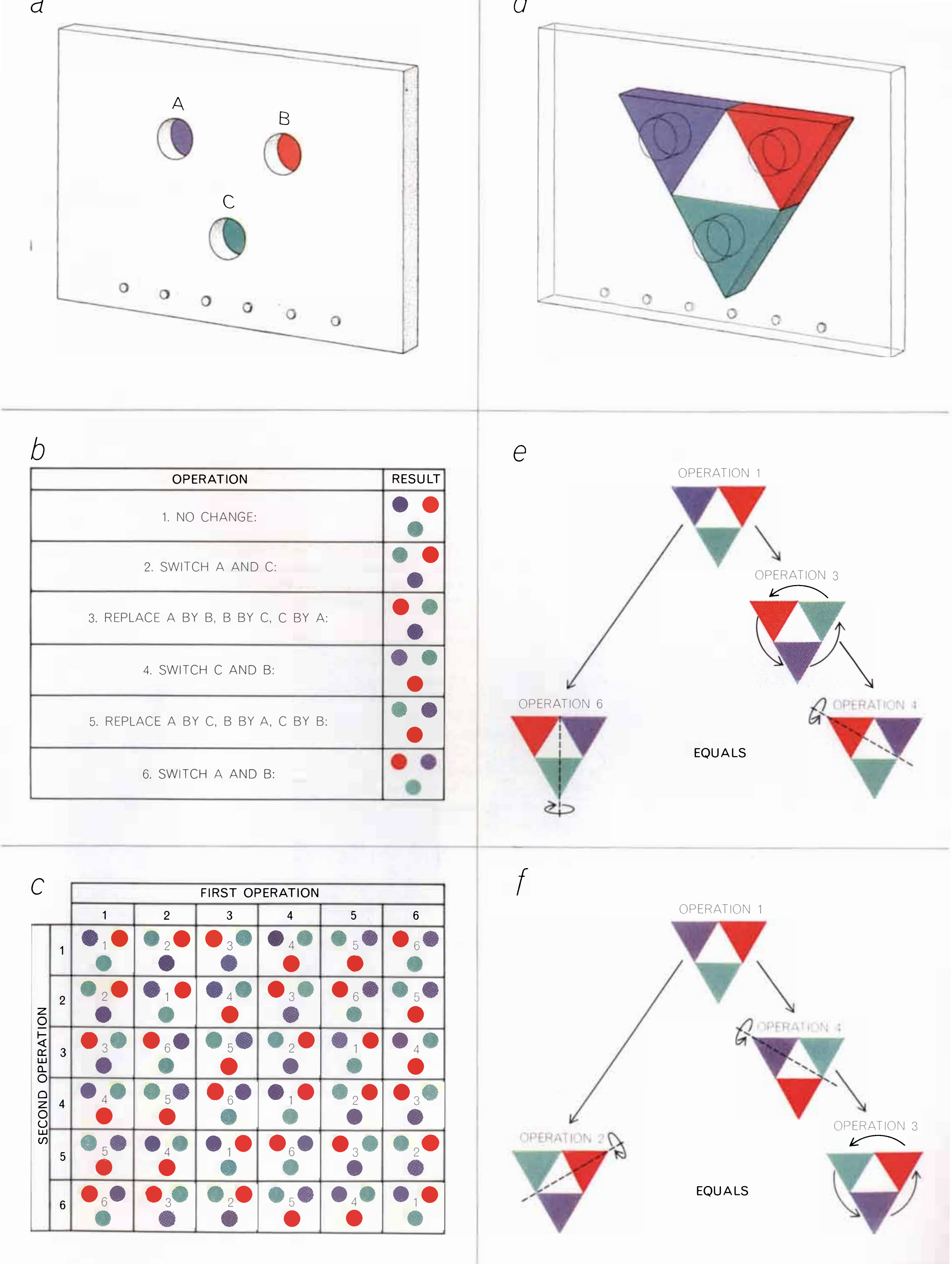
The older algebra provided a versatile tool for many practical and scientific enterprises. Archaeological evidence indicates that algebraic formulas for finding the volumes of cylinders and spheres may have been used in ancient Egypt to compute the amount of grain a peasant owed in taxes to the central government. Before the advent of the calculus and celestial mechanics in the late 17th century, algebra and geometry were the twin mainstays of all astronomical reckoning. The essential operations of the older algebra are familiar to anyone who has studied the traditional curriculum at a good high school. In fact, a person who has mastered elementary algebra and geometry in high school is in a good position to learn without any great mental readjustment most of the mathematics discovered prior to 1800. The reason is that mathematics before 1800 was essentially concerned with only two commonsense ideas, number and shape.

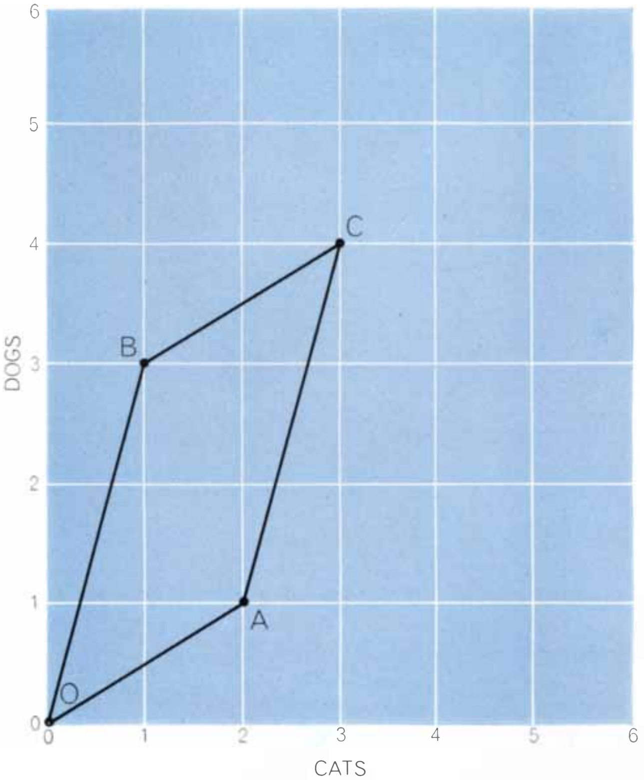
Early in the 19th century this whole view of mathematics began to change. Two new ideas were introduced that profoundly expanded the domain of mathematics. The first was that mathematics did not need to restrict itself to numbers and shapes but could deal effectively with anything (although "anything" often continued to be related in some way to numbers and shapes). The second idea carried the process of abstraction a step further: mathematics could at times be regarded merely as a logical procedure, having no relation to anything in particular.

Scientists, as distinct from mathematicians, are attracted to the first idea; it suggests that mathematics may have a much wider sphere of application than had ever been imagined in the earlier epoch. The second idea appeals more to pure mathematicians, who have come to regard mathematics simply as the study of beautiful patterns. There is no real conflict between the two viewpoints. A pattern conceived for its beauty by a pure mathematician may well prove to fit some aspect of the physical world; conversely, some of the mathematical patterns discovered in nature by the scientist have turned out to be remarkably beautiful.

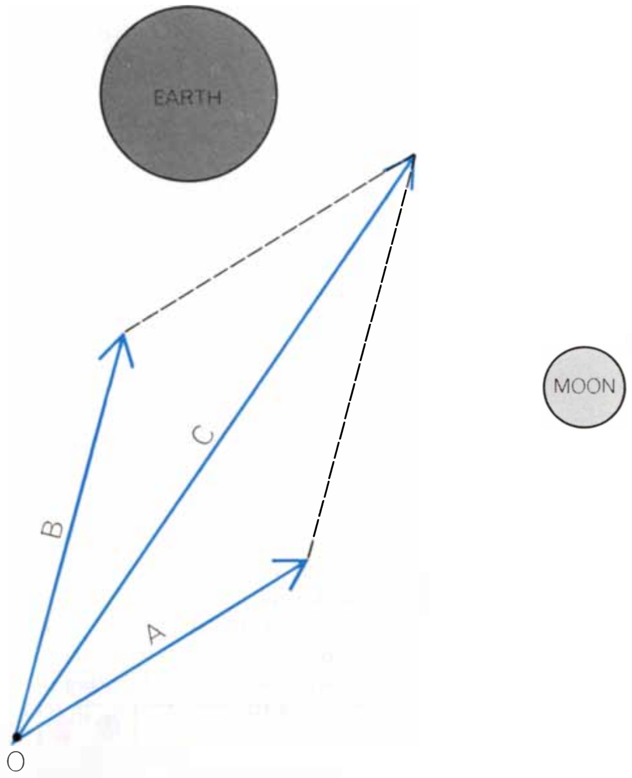
It would of course be beyond the scope of any one article to attempt to trace all or even most of the effects of these two new ideas on the course of modern algebra. For one thing, algebra itself has become so compartmentalized that each separate branch would have to be treated more or less in isolation. On the other hand, isolated snippets of information are unlikely to satisfy the general reader, who would remain ignorant of the overall framework that gives these fragments their significance. The only satisfactory solution seems to be to cut off a rather large chunk of one

ALGEBRA OF GROUPS is applied to a hypothetical phenomenon in the physical world in the illustration on the opposite page. Imagine a wall with three holes through which three different colors appear (*a*). The order of the colors can be changed in six different ways (*b*) by pushing the six buttons at the bottom of the wall. These changes, or operations, occur in a pattern that fulfills the four basic requirements of a group: (1) If one operation is followed by another, the result is the same as if one operation were performed alone. (2) Any series of operations follows the associative rule, which can be expressed in symbols as $x(yz) = (xy)z$. (3) For every operation there is an inverse operation. (Operations 3 and 5 are the inverses of each other; the other operations are their own inverses.) (4) There is one identity operation, which leaves the original order unchanged (operation 1). The consequences of any pair of operations appear in table *c* (the table is called a matrix). The "simplest" physical explanation of the observed phenomenon would be to postulate the existence of a suitably colored triangular block behind the wall (*d*). Keeping the same orientation in the plane, the triangle can be rotated or turned over in only six different ways, which correspond exactly to the six color arrangements seen through the holes in the wall. This particular group is called a finite, noncommutative group: finite because the number of operations is finite, noncommutative because the sequence in which operations are performed can affect the outcome (xy does not necessarily equal yx). A demonstration of the noncommutative nature of the group is given in *e* and *f*, which show that operation 3 followed by operation 4 does *not* produce the same result as operation 4 followed by operation 3.

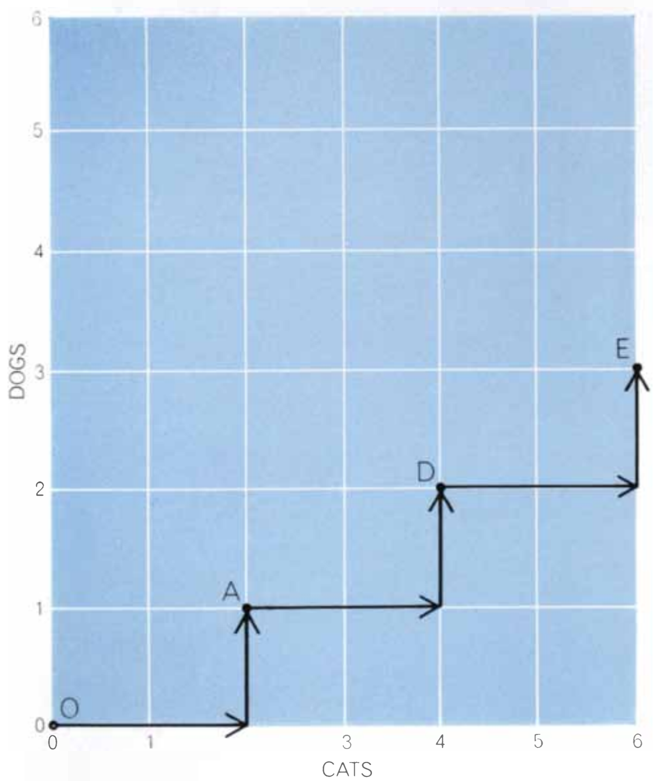

a
A
B
C
d
b
OPERATION
RESULT
1. NO CHANGE:
2. SWITCH A AND C:
3. REPLACE A BY B, B BY C, C BY A:
4. SWITCH C AND B:
5. REPLACE A BY C, B BY A, C BY B:
6. SWITCH A AND B:
c
FIRST OPERATION
1 2 3 4 5 6
SECOND OPERATION
1: 1 2 3 4 5 6
2: 2 1 4 3 6 5
3: 3 6 5 2 1 4
4: 4 5 6 1 2 3
5: 5 4 1 6 3 2
6: 6 3 2 5 4 1
e
OPERATION 1
OPERATION 3
OPERATION 6
OPERATION 4
EQUALS
f
OPERATION 1
OPERATION 4
OPERATION 2
OPERATION 3
EQUALS



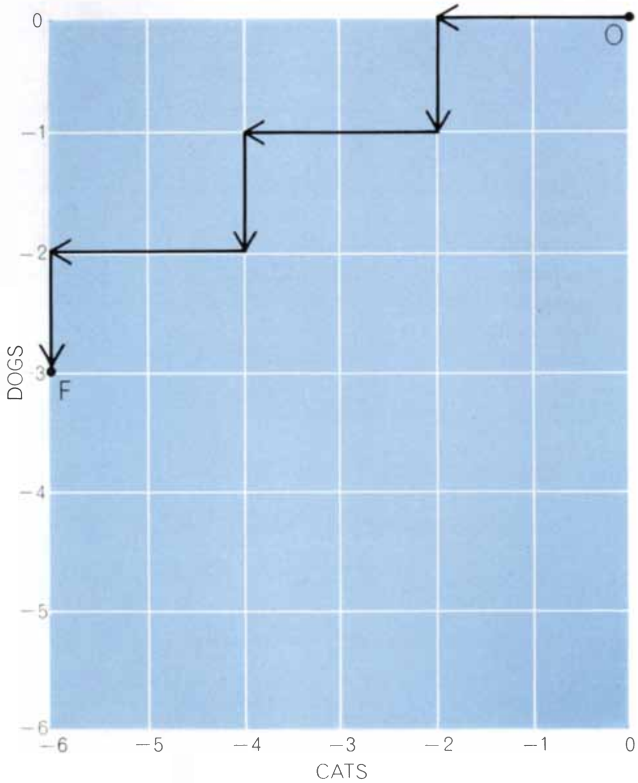
ADDITION involved in statement "Two cats and one dog + one cat and three dogs = three cats and four dogs" can be illustrated by means of a parallelogram. Since point *C* represents the sum of the quantities represented by *A* and *B*, one can write $C = A + B$.



VECTOR ADDITION also involves construction of a parallelogram. A diagonal of the parallelogram (*Vector C*) represents the total gravitational force exerted on observer at *O* by the earth (*Vector B*) and the moon (*Vector A*). Again one can write $C = A + B$.



MULTIPLICATION involved in statement $E = 3A$ can be interpreted in several different ways. If *A* signifies two cats and one dog, *E* equals six cats and three dogs. Geometrically point *E* lies in the same direction from *O* as *A* but is three times farther away.



NEGATIVE NUMBERS are needed to represent a debt that obliged us, say, to deliver six cats and three dogs (*F*). The transition from *O* to *F* could also represent a three-stage journey, each stage of which has the same specification: two units west and one south.

branch of modern algebra and explain it in detail. The reader will then be able to gain some idea of the general direction in which algebra is moving from a consideration of how this particular branch has developed in the years since 1800. I have chosen to devote most of this article to a detailed discussion of vector algebra and matrix algebra, two subjects that are just beginning to find their way into the high school curriculum. Both subjects have already made significant contributions both to pure mathematics and to science. Finally, I shall sketch more briefly several of the other departments of modern algebra in which interesting and significant work is in progress.

The new ideas about algebra that emerged at the beginning of the 19th century grew naturally out of the older algebra. One stimulus was the concept of the square root of minus one, customarily denoted by i, which was used to solve a wide range of problems in the 17th and 18th centuries but which no one seemed able to explain satisfactorily as a number. In the early 19th century two different solutions to this dilemma were proposed. The first employed what is called the abstract method, in which i was interpreted as a series of rather arbitrary operations on pairs of numbers [see "The Foundations of Mathematics," page 112]. The second solution gave i a concrete interpretation, identifying it with the geometrical operation "Rotate through a right angle in a plane."

Both explanations suggested further exploration. Since excellent dividends had accrued from the introduction of i into the procedures of elementary algebra, might it not be profitable to bring in a few more meaningless symbols? The rules governing these symbols could then be tailored to meet the demands of the situation. If i was interpreted as a rotation in a plane, why not consider rotations in three-dimensional space and see if these could contribute anything to algebra? This whole line of inquiry led eventually to the discovery of quaternions in 1843 by William Rowan Hamilton. In quaternion algebra two new symbols, i and j, were introduced, with the rules $i^2 = -1$, $j^2 = -1$ and the surprising $ji = -ij$ [see "Number," page 50].

Many prominent British mathematicians of Hamilton's day were completely swept off their feet by the discovery of quaternions, which they regarded as the last word in mathematics and the ideal method for solving most algebraic problems. Actually quaternions were a first rather than a last word. A barrier had been crossed; an algebra had been developed that disregarded several of the basic conventions of the older algebra. Mathematicians soon began to look for other ways in which ordinary numbers could be supplemented by new symbols to produce what came to be known as "hypercomplex numbers." Eventually the question was bound to arise: Why start with ordinary numbers? Why not consider any collection of symbols and lay down rules for combining them? The word "algebra" was gradually being expanded to include any system of handling symbols according to prescribed rules. Anyone was now free to invent his own algebra. Of course this did not mean that anyone who did so earned a right to immortality; the problem has always been to invent a system that yields interesting and fruitful results and contributes significantly to the rest of mathematics and to science.

Algebra thus ceased to be about anything in particular, although this did not rule out the possibility that a particular system of algebra could be applied to a particular topic. Such a topic would then be said to have an algebraic aspect.

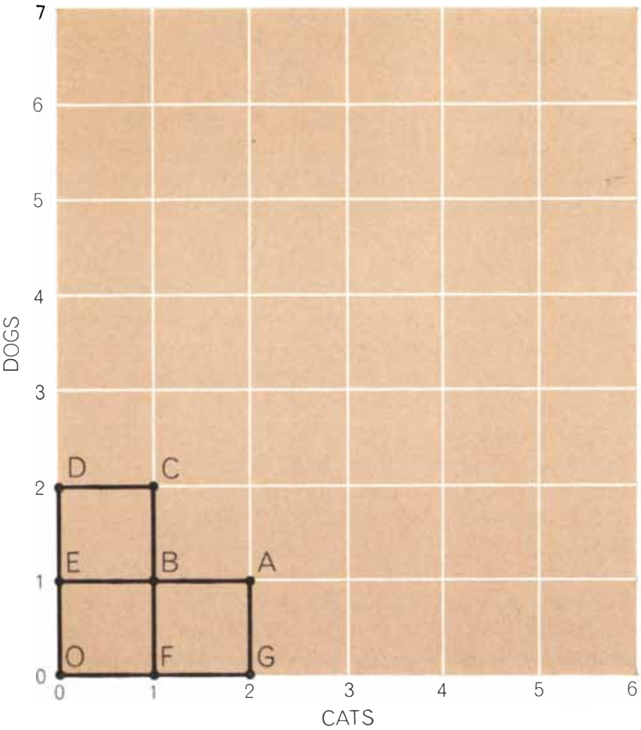
In elementary algebra the symbols stand for numbers and the sign + indicates that the numbers are to be added. It may surprise the reader to learn that after algebra had ceased to be exclusively occupied with numbers the sign + continued to be used. How can one add things that are not numbers? The explanation is that in modern algebra the sign + does not indicate that addition is being carried out in any real sense. It merely means that some operation is being carried out according to rules that remind a mathematician of the rules for addition. The resemblance is one of pattern, not of content.

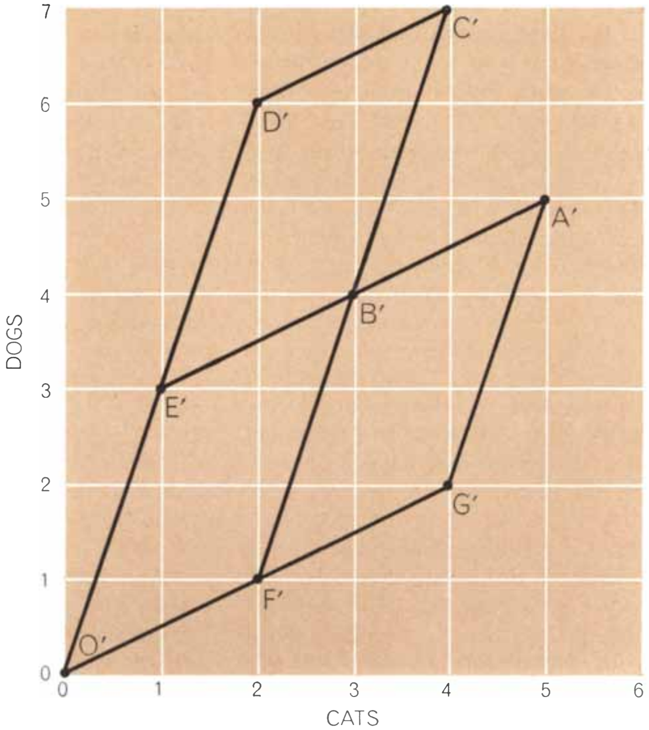
Let us now consider an example of this broadening of the meaning of the sign +. We shall begin with a situation clearly related to addition and from it derive another situation in which the connection with addition is much less apparent.

"Adding" in arithmetic is generally associated with the idea of "putting to-

DOGS \ CATS	0	1	2	3	4	5	6
0	0 / 0	2 / 1	4 / 2	6 / 3	8 / 4	10 / 5	12 / 6
1	1 / 3	3 / 4	5 / 5	7 / 6	9 / 7	11 / 8	13 / 9
2	2 / 6	4 / 7	6 / 8	8 / 9	10 / 10	12 / 11	14 / 12
3	3 / 9	5 / 10	7 / 11	9 / 12	11 / 13	13 / 14	15 / 15
4	4 / 12	6 / 13	8 / 14	10 / 15	12 / 16	14 / 17	16 / 18
5	5 / 15	7 / 16	9 / 17	11 / 18	13 / 19	15 / 20	17 / 21
6	6 / 18	8 / 19	10 / 20	12 / 21	14 / 22	16 / 23	18 / 24

MATRIX is used to find the yields of a hypothetical animal "banking scheme" that offers the following interest rates: For each cat deposited now, one will be entitled to two cats and a dog a year from now; for each dog deposited now, one will be entitled to a cat and three dogs a year from now. The light numbers signify deposits; darker numbers signify yields.





ANIMAL BANKING SCHEME used to construct matrix on preceding page can also be shown graphically. Lettered points in graph at left signify specific deposits; corresponding points in graph at right represent yields from these deposits. The shift from squares representing deposits to parallelograms representing yields is characteristic of any algebraic "banking scheme" of this general type.

gether." In cooking one speaks of adding water to a mixture. Most people would find no difficulty in following the addition involved in the statement: "Two cats and one dog + one cat and three dogs = three cats and four dogs." They would regard this as a legitimate use of the sign +.

The operation of addition in the foregoing statement can be illustrated graphically [*see illustration at top left on page* 72]. Here cats are indicated horizontally and dogs vertically. Point *A* represents two cats and one dog; that is, *A* is two units across and one up from the point of origin (*O*). By the same token point *B* corresponds to one cat and three dogs. Since point *C* represents the sum of these two pairs of numbers (three cats and four dogs) we might be led to write $C = A + B$.

Suppose we were to show this graph to someone who did not know anything about the cat-and-dog story and ask the person to describe what he saw. The chances are he would say: "There is some graph paper with four points, *O, A, C* and *B,* marked on it so as to form a parallelogram." Thus it is possible to give a purely geometric description of how to get point *C* from *A* and *B:* one simply chooses *C* so that together with *O, A* and *B* it completes a parallelogram.

Clearly the graph derived from the statement "Two cats and one dog + one cat and three dogs = three cats and four dogs" can be approached from at least two different viewpoints. In the context of the cat-and-dog story the situation clearly involves addition from the start and the graph is simply a device to illustrate addition. From a purely geometric point of view *C* is the point needed to complete the parallelogram *OACB*. In the latter case it would seem highly unnatural to write $C = A + B$, or indeed to expect any connection at all with algebra. But since the graph itself remains one and the same thing no matter how we view it, it is clear that the geometry of the parallelogram must have a strong algebraic aspect related in some way to addition.

Anyone who has studied mechanics or electricity and magnetism is familiar with the fact that nature often uses the parallelogram as a means of addition. In the illustration at top right on page 72 the letter *O* represents an observer somewhere in space. The line *OB* represents the gravitational pull of the earth, that is, the force that would act on the observer if the earth were the only massive body in his vicinity. Similarly, the line *OA* represents the gravitational pull of the moon by itself. In fact, both the earth and the moon are pulling at the observer simultaneously, so that to calculate the actual force acting on the observer we must combine, or "add," the effects the earth and the moon would produce separately. This combined force can be represented by the line *OC,* which turns out to be a diagonal of the parallelogram *OACB*.

If we now wished to take into account the force exerted by the sun, we would have to "add" the pull of the sun to the force represented by *OC,* again employing the parallelogram technique. At this point one of the important analogies to the addition of ordinary numbers becomes apparent. If one adds any three numbers, the sequence in which one adds them is immaterial. If one has to pays bills, say, of \$3, \$5 and \$6, one cannot economize by paying them in some special order. In whatever order one pays one is bound to part with the same total sum: \$14. This is called the commutative rule of addition. (Strictly speaking, the associative rule of addition, which in symbols states that $a + (b + c) = (a + b) + c$, is also involved here.) Mathematicians would find it misleading to use the sign + for any procedure in which the sequence of the operations affected the final outcome.

Obviously the sequence of operations does not affect our gravitational problem, since the observer at *O* is simultaneously acted on by the earth, the moon and the sun. In our computation of the total gravitational effect on the observer we can first combine the pull of the earth and the pull of the moon and then "add" this result to the pull of the sun. We could just as well start by combining the effects of the sun and the moon and then "add" this sum to the pull of the earth. If the two procedures were to result in different answers, our purely geometric technique would obviously be unsatisfactory. The physics of the situation demands that the commutative and associative rules apply. Insofar as our procedure for combining forces must obey these rules, it resembles the addition of ordinary numbers.

The parallelogram technique described in the last two examples is known as vector addition. In both examples the lines OA, OB and OC are called vectors and are usually drawn as arrows.

Another way to interpret the cat-and-dog graph is in terms of journeys. The distance from O to A corresponds to a journey of two units to the east and one unit to the north; similarly, the distance from O to B corresponds to a journey of one unit to the east and three units to the north. If we combine these journeys, traveling first two units east and one north and then one east and three north, we find that in all we have journeyed three units east and four north, or the distance from O to C. The story is different, but once again we find that C can be interpreted as the sum of A and B.

We have seen that several different meanings can be attached to the expression $A + B$. What are some of the ways in which we can interpret the expression $3 \times A$, or $3A$?

If A signifies two cats and one dog, there should be little doubt about the meaning of $3A$; the product is obviously six cats and three dogs. In the illustration at bottom left on page 72 this product is indicated by the point E; thus we can write the equation $E = 3A$. Geometrically we see that E lies in the same direction from O as A does, but that it is three times farther away. In addition to this purely geometric interpretation of $E = 3A$, the interpretation as journeys is again helpful; the journey from O to E can be broken into three stages—O to A, A to D and D to E—each of which has the same specification: two units east and one north.

Let us now introduce the symbols c and d as abbreviations for cats and dogs respectively. We have seen that the expression $6c + 3d$ can be interpreted in at least three different ways: (1) in its original meaning of six cats and three dogs, (2) as a way of specifying the position of the point E in the graph at bottom left on page 72 and (3) as describing the stepwise journey from O to E in the same illustration.

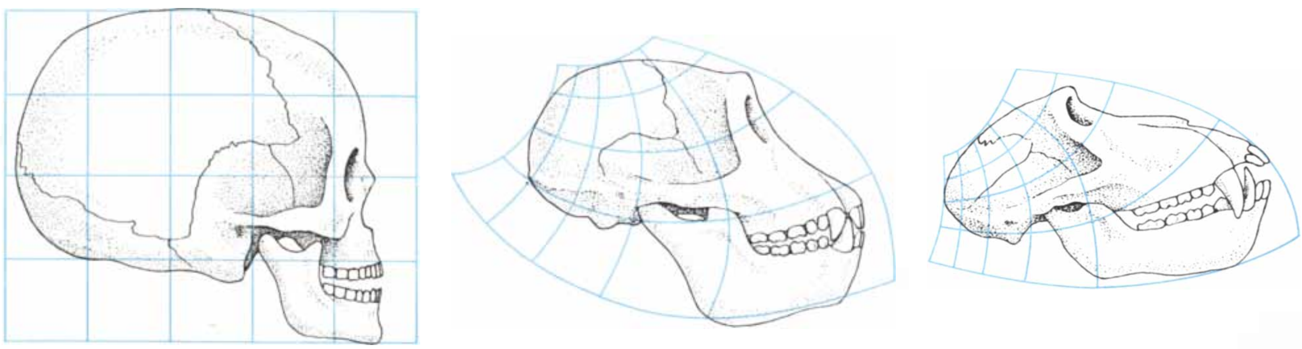
Our symbolism still suffers from one restriction. We have no difficulty in writing $6c + 3d$ for the overall journey from O to E, but what about a journey in the opposite direction? For this we need negative numbers; thus the expression $-6c - 3d$ could be interpreted as a journey of six units west and three south. The same expression could also be used to specify the position of the point F in the graph at bottom right on page 72. Or it could represent minus six cats and minus three dogs, that is, a debt that obliged us to deliver six cats and three dogs.

We are now able to specify the position of any point in a plane by a pair of algebraic symbols such as $6c + 3d$. Moreover, we can translate a number of geometric constructions into simple algebraic operations on these symbols; for example, drawing a parallelogram corresponds to addition. Many theorems in elementary plane geometry can be proved without reference to geometric constructions simply by carrying out the appropriate algebraic calculations with c and d.

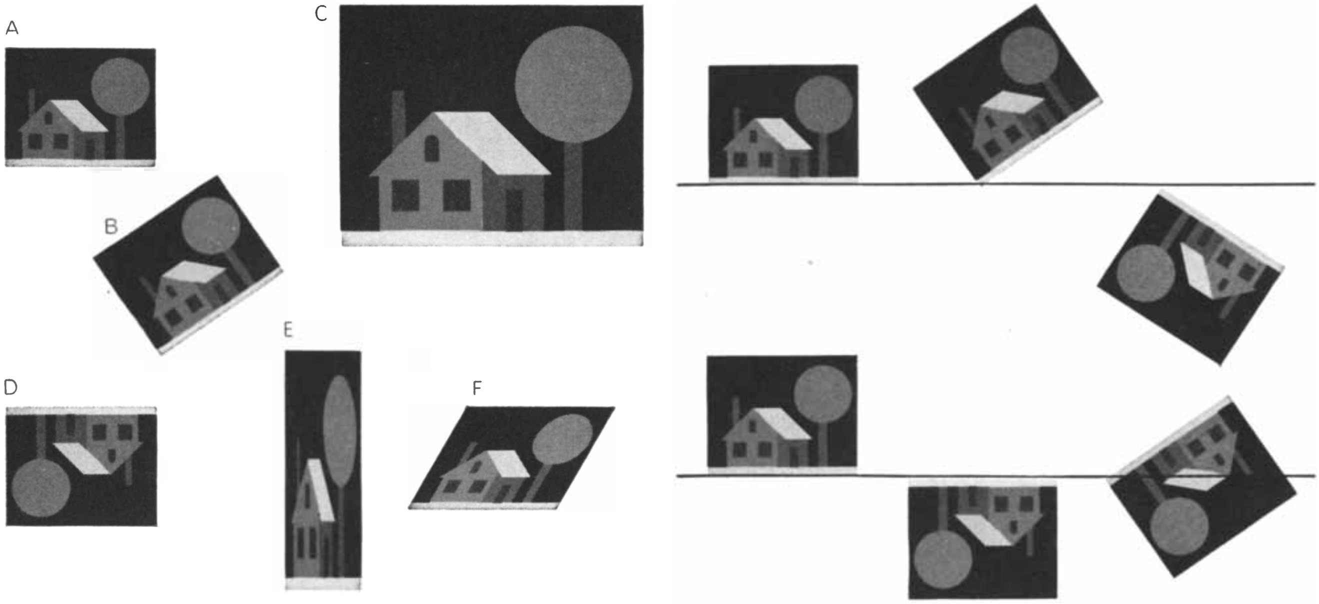
The illustration on the next page shows a few geometric distortions that can be performed on a rectangular picture. In B the original picture (A) is tilted, or rotated counterclockwise around its bottom left corner; in C it is enlarged; in D it is reflected as in a mirror; in E it is stretched vertically and shrunk horizontally; in F it leans to the right. Do these operations have an algebraic aspect? Can we add and multiply them? For instance, can we add a rotation to a reflection and state the procedure as $B + D$? It may appear to be unlikely and yet it can be done. Such operations have in fact a remarkably simple algebraic aspect.

In order to illustrate the preceding statement let us return to the cat-and-dog story. Imagine a society in which cats and dogs represent wealth, as sheep and cattle do in some primitive societies. In our hypothetical society a bank might offer the following interest rates: For each cat deposited now, one will be entitled to two cats and a dog a year from now; for each dog deposited now, one will be entitled to a cat and three dogs a year from now. In symbols this could be expressed $c \rightarrow 2c + d$; $d \rightarrow c + 3d$. (The arrow signifies "yields.") It is fairly easy to work out the yield for a deposit of any given number of cats and dogs. A deposit of a cat and a dog, for instance, would yield three cats and four dogs ($c + d \rightarrow 3c + 4d$). A deposit of two cats and a dog, on the other hand, would yield five cats and five dogs ($2c + d \rightarrow 5c + 5d$). The yields for various other combinations of cat-and-dog deposits are indicated in the table on page 73. A table of this type is called a matrix and the branch of mathematics that employs such tables is called matrix algebra.

Let us now translate all our symbols



COMPARISON OF SKULLS of a man (*left*), a chimpanzee (*center*) and a baboon (*right*) can be said to have an algebraic aspect. As in the case of the animal banking schemes, the rectangular coordinate system over the human skull can be progressively distorted by means of the appropriate algebraic operations. Diagrams of this type have been used in a wide range of morphological studies.



SOME GEOMETRIC DISTORTIONS that can be performed on a rectangular picture (*A*) are depicted at left. The distortions all have an algebraic aspect: distortion *F* corresponds to the algebraic banking scheme that converts squares to parallelograms; the other distortions correspond to similar banking schemes. Any combination of distortions is noncommutative; the illustration at right shows that rotating a reflected picture does *not* produce the same result as reflecting a rotated picture (in symbols, $B + D \neq D + B$).

into geometry. In the graphs on page 74 point A represents a deposit of $2c + d$ and A' represents the yield of this deposit, or $5c + 5d$. Some other points have been marked in; B, for instance, represents the deposit $c + d$ and B' the corresponding yield $3c + 4d$. By the same token C' represents the yield that would result from the deposit C, D' the yield from the deposit D and so on. It is apparent that the points in both graphs are arranged in orderly patterns: the original points (A, B, C, D, E, F and G) form squares and the corresponding primed points (A', B', C', D', E', F' and G') form parallelograms.

This shift from squares to parallelograms is not fortuitous. In fact, any algebraic "banking scheme" of this general type will produce graphs in which squares representing deposits become parallelograms representing yields. Conversely, any geometric distortion that converts squares to parallelograms in this manner can be produced by means of a suitable algebraic "banking scheme." The distortion designated F in the illustration above is an example of this type of square-to-parallelogram distortion. All the other distortions shown in the same illustration correspond to similar "banking schemes." Thus the scheme $c \rightarrow 2c$, $d \rightarrow 2d$ would yield an enlargement corresponding to distortion C, whereas the scheme $c \rightarrow \frac{1}{2}c$, $d \rightarrow 2d$ would yield distortion E. Slightly more complicated algebraic schemes are required to produce the distortions shown on the preceding page, in which the skulls of a man, a chimpanzee and a baboon are compared geometrically. Comparative diagrams of this type have been used to study a wide range of morphological characteristics, which can be said to have an algebraic aspect.

Is it possible to "add" two such banking schemes? Suppose in our hypothetical society three banks, X, Y and Z, were to offer three different interest rates; what then could we understand by $X + Y = Z$? It could mean that scheme Z yields as much as schemes X and Y put together. By calculating the yield of scheme X for a given deposit and then adding this quantity to the calculated yield of scheme Y for the same deposit, we can determine what scheme Z would yield for that deposit. Similarly, we could interpret the equation $A = 3B$ as meaning that the yield of scheme A is equal to three times the yield of scheme B for the same deposit.

By combining the last two ideas we can now interpret an expression such as $4A + 5B$. The yield of this combined scheme, for any deposit, would equal four times the yield of scheme A added to five times the yield of scheme B.

It is possible to go on to discuss the multiplication of "banking schemes." The expression AB would indicate that one first invested in scheme B and then reinvested the yield in scheme A. That it is not unreasonable to call this process "multiplication" can be seen from the following example. Suppose the effect of scheme B is to double one's deposit and the effect of scheme A is to treble the deposit. If one invests first in scheme B and then reinvests the proceeds in scheme A, the total effect is to multiply the initial investment by six. Since six equals three times two, it is not unreasonable to associate reinvestment with multiplication.

What do all these examples prove? First, the various cat-and-dog stories have demonstrated that algebra can be made to apply to a number of geometric situations that at first sight appear to be in no way related to algebra. Second, the algebra involved is remarkably straightforward; most of the expressions we have used have been simple ones, such as $6c + 3d$, that are familiar from elementary algebra. Third, the broadening of the meaning of the word "algebra" has many important implications. We have been talking about the way in which forces are exerted, the way shapes are distorted under stress, the way things change their position—all topics of obvious importance in science and engineering. There are many less obvious applications both in science and in higher mathematics.

In the preceding discussion of vector and matrix algebra we have dealt largely with models; that is, we have examined certain actual situations and operations—collections of animals, journeys, investments, rotations and reflections—and have found in these situations and operations elements that reminded us of the addition and multiplication of ordinary numbers. The requirement of "reminding us," however, is somewhat vague. How closely must something remind us of addition to deserve being represented by the sign +? Clearly more specific rules are needed if there is not to be a great deal of confusion about symbolism.

We have already mentioned a few of the requirements for an operation to be recognized as a generalization of addition. The commutative rule, for example, states that whatever objects a and b might stand for and however complicated the operation of combining them might be, we are always entitled to expect that $a + b$ will mean the same as $b + a$.

In ordinary arithmetic multiplication is also commutative: 3×4 always means the same as 4×3. In fact, the properties of multiplication resemble those of addition very closely. It is only by virtue of this fact that we are able to construct tables of logarithms, which convert multiplication to addition. Since it would be wasteful to have two symbols with the same implications, the convention has arisen among mathematicians that the sign + may be used only for commutative systems, but the sign $\times$ need not carry this restriction. In some branches of algebra $a \times b$ and $b \times a$ may denote the same object, but they are not obliged to. When $a \times b$ and $b \times a$ have different meanings, the algebra is said to be noncommutative.

We do not have to look far for an example of noncommutative algebra: our previous discussion of matrix algebra provides several. In the illustration on the opposite page the sequence in which the operations "rotate" (B) and "reflect" (D) are performed on the original picture can be seen to have an effect on the outcome. Reflecting the rotated picture does *not* produce the same result as rotating the reflected picture; in symbols, $B + D$ does *not* equal $D + B$. It is also possible to construct various "animal-investment banking schemes" in which the sequence of investment and reinvestment affects the final outcome. (This is not the case in ordinary banking, in which reinvestment is always commutative.)

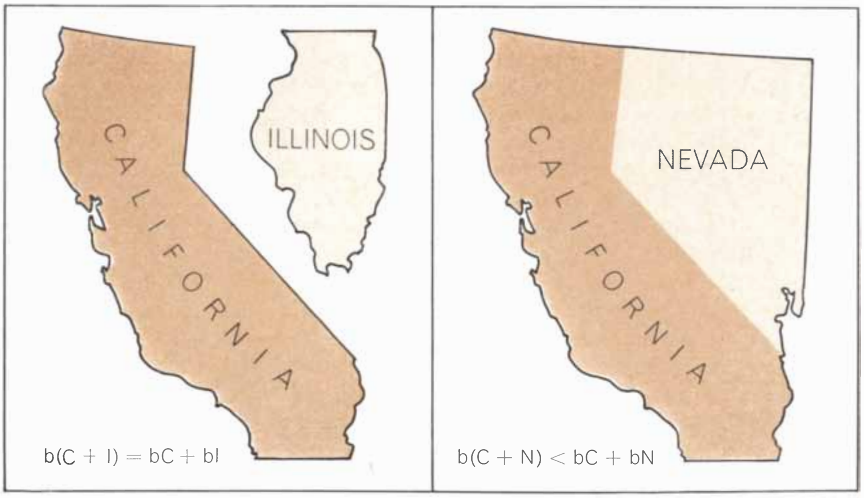
Another important property of addition and multiplication in ordinary arithmetic is that these operations are associative. In other words, if one has to work out $3 + 4 + 5$, it does not matter whether one obtains the answer by considering $7 + 5$ or by considering $3 + 9$. In the case of multiplication the associative rule requires that $3 \times 4 \times 5$ can be found by means of 12×5 or by means of 3×20. There is perhaps some subtlety involved in this concept. The reader may ask: Are we not saying simply that the system is commutative, that the order in which one adds or multiplies is immaterial? The distinction may be elucidated by considering the phrases "fat-cattle merchant" and "fat cattle-merchant." The order of the words is the same but the meaning is different. What has been changed is the manner in which the words have been grouped together. The associative property deals with the effect of punctuation, not of order. That the two properties are genuinely distinct can be further appreciated by considering the fact that matrix algebra is not commutative but is associative. The majority of algebraic systems that have proved productive so far have been associative, although the theory of nonassociative algebra has recently attracted the attention of some mathematicians.

A third basic property of ordinary arithmetic is expressed by the distributive rule, which in symbols states that $a(b + c) = ab + ac$. The distributive property has a way of turning up in unexpected situations. For example, the boundary of the area covered by California and Illinois is the boundary of California added to the boundary of Illinois. It is not difficult to see a connection between this statement and the equation $b(C + I) = bC + bI$, in which b stands for "the boundary of" and C and I are the initials of the states involved. It is interesting to note that the distributive rule ceases to apply if the states in question have a common boundary [*see illustration below*].

The concept of boundaries occurs in topology and in certain areas of the calculus. What is important to realize here is that algebra can play a role inside other branches of mathematics, and not merely because numbers are involved in these branches. In fact, the branches of mathematics interact to a surprising degree.

In algebra the word "field" is used to describe a system that closely resembles ordinary arithmetic. The operations of addition, subtraction, multiplication and division occur in a field and are much like the corresponding operations in arithmetic. For instance, there is in every field an element designated by the letter "O" that resembles zero (it makes no difference if you add O) and an element I that resembles one (it makes no difference if you multiply by I). A large variety of fields exist. The tables on the next page specify a field in which there are only four elements: O, I, A and B. Within this system we operate by essentially the same rules as we do in arithmetic and elementary algebra; in it the commutative, associative and distributive properties all hold. The possibility of a field with a finite number of elements was discovered by the French mathematician Évariste Galois in 1830. This particular field is called a Galois field with four elements.

Take any formula from high school algebra and you will find that it remains true in the Galois field. For example, elementary algebra leads us to believe that $A + B$ multiplied by $A - B$ should



DISTRIBUTIVE RULE of ordinary arithmetic states in symbols that $a(b + c) = ab + ac$. In this example of the distributive rule the boundary of the area covered by California and Illinois is the boundary of California added to the boundary of Illinois; in symbols, $b(C + I) = bC + bI$. The distributive rule does not apply in the case of California and Nevada, which have a common boundary. Since the concept of boundaries occurs both in topology and in calculus, algebra is brought into these branches in cases that do not involve numbers.

	O	I	A	B
O	O	I	A	B
I	I	O	B	A
A	A	B	O	I
B	B	A	I	O

	O	I	A	B
O	O	O	O	O
I	O	I	A	B
A	O	A	B	I
B	O	B	I	A

GALOIS FIELD is an abstract algebraic system containing only four elements, *O*, *I*, *A* and *B*, which can be added, subtracted, multiplied and divided by essentially the same rules that hold in arithmetic and elementary algebra. Table at left is for addition; table at right, for multiplication. Galois fields have recently been applied to error-free transmission of information by high-speed machines.

be the same as $A^2 - B^2$. This is also true for the Galois field. Using the tables to work out these two results, you will find that both give the same answer: *I*.

At this point we have reached a completely abstract stage. There is no suggestion that *O*, *I*, *A* and *B* have any meaning. We are no longer talking about collections of animals or movements of bodies. We have simply found a pattern that has interesting analogies with the patterns of ordinary arithmetic. A very pure mathematician would say that this is the whole object of mathematics—to discover beautiful and interesting patterns. An applied mathematician, a scientist or an engineer would be interested in knowing if this pattern is one that occurs in nature and can therefore be given an interpretation and an application. Although conceived as an abstract mathematical exercise, Galois fields have recently found a rather unexpected application: they have been studied in connection with error-free codes for the transmission of information by high-speed machines.

In a field we can add, subtract, multiply and divide (except that division by *O* is barred). Not all algebraic systems have as comprehensive a list of operations. In a ring, for instance, we can add, subtract and multiply but not necessarily divide. A familiar example of a ring is the whole numbers, both positive and negative. If a person were conversant only with numbers such as ...−4, −3, −2, −1, 0, 1, 2, 3, 4..., he would be able to solve any problem that involved the addition, subtraction or multiplication of these numbers. If asked to divide 3 by 4, however, he would be powerless.

Even more restricted than a ring is the concept of a group. When we say that some system is a group, we promise only the existence in it of one operation, which can be thought of as a kind of generalized multiplication. This operation must be associative: an expression such as *XYZ* must have a definite meaning regardless of punctuation. The group must also contain an element *I* that resembles the number one in arithmetic. Furthermore, division must be possible. An example of a group with six elements appears on page 71.

To qualify as a group an algebraic system must pass a surprisingly small number of tests. It is therefore all the more remarkable that such an elaborate theory of groups has been developed, with such widespread ramifications in higher mathematics and physics [see "Mathematics in the Physical Sciences," page 128].

Many new algebraic systems have arisen out of particular problems in other branches of mathematics. Near the end of the 19th century the Norwegian mathematician Sophus Lie completed a comprehensive classification of differential equations in calculus. Certain patterns in this classification system have since been extracted and now constitute a separate subject, known as Lie groups. Similarly certain problems in topology have led to the new subject of homological algebra, which has also proved to have important applications outside of topology.

In the late 1840's the English mathematician and logician George Boole developed a system of symbolic logic in which the propositions of Aristotelian logic were reduced to equations that were closely analogous to those in elementary algebra. The system follows many of the rules of ordinary arithmetic, including the commutative, the associative and the distributive rules. Boolean algebra has recently been applied to the design of telephone circuits and electronic computers.

Algebra, like every other branch of mathematics and science, continues to proliferate with the vitality and expansiveness of a tropical forest. It is a difficult situation. To know everything is clearly impossible, yet each specialist assures you that you must know the particular part of algebra he finds interesting. The scientist who uses mathematics should be aware that much new mathematical knowledge is being discovered; nearly all of it will be irrelevant to his own research, but he should keep his eyes open for the small piece that may be of great value to him.

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With first and second derivatives cheerfully positive here for many decades, some of us may currently be working too hard.

In the interests of good sense, Jim Bruce has been given the task of investigating to what extent the plight of the overbusy ones might complement that of engineers and engineering physicists elsewhere who have likewise been working very hard since college or graduate school but who are now encountering a shrinkage of work load to a point where they are enjoying themselves less.

By telling Jim Bruce of their problem they might solve his!

James S. Bruce is Director, Business and Technical Personnel Department, Eastman Kodak Company, Rochester, N. Y. 14650. We are an equal-opportunity employer welcoming résumés and statements of qualifications.

A more specific part of Jim's task—but only part—is to extend invitations to join a brand new research staff that concentrates in the broad field of engineering physics and its application to certain apparatus. Whoever thinks we refer here only to apparatus such as we advertise on TV seems to have missed the kind of contacts and experience that would justify further time spent on this matter.

Jim is prepared to forgive expressions of mild surprise that the same old-line company that provides lifetime professional careers to such settled types as biochemists and plastics engineers has similar propositions for stress-and-vibration specialists, thermodynamicists and systems analysts who would feel at home with our 7044 and our other scientific-type computers.

Fairly chronic is Jim's need to know an ever-widening circle of opticists who have acquired polish in such subjects as modulation transfer functions and would like to practice here at the headquarters for these subjects. Now he also has to find electrical engineers, mechanical engineers (who lean more toward camera shutters than earth-movers), and instrumentation specialists—colleagues to share the load with our design, reliability, and quality-control engineers who wish to renew ties with family and neighbors in the prosperous, pleasant, and cultured Western New York countryside.

The other way of maintaining strength continues basic with us. Jim and his men are about to fan out to 130 campuses or so for the next few months seeking vigorous scientists and engineers who want to start with us right from the winning of their degrees, B.S., M.S., or Ph.D. We shall appreciate any kind word that faculty and placement directors would care to put in on our behalf with candidates who give prospect of achieving happiness with us. That covers a broad spectrum. It ranges from a few who are so theoretical that they scarcely know which way to point an INSTAMATIC Camera to a few who are so practical that they can't draw an energy-level diagram without dollar signs. Each gets his reward.

Why no infrared opticist who is in touch with the situation asks "What?" any more but "Which?" when someone says, "IRTRAN"

In recent months it has become apparent that the five types of KODAK IRTRAN hot-pressed polycrystalline optical materials lead the field. This is perhaps attributable largely to the success of our program to eliminate absorption bands from the IR ranges which each is intended to transmit. At .080″ thickness, for instance, all we still have to cope with are the H_2O band at 2.7μ and one other at 5μ, both in IRTRAN 1 Material. Get the latest data from Eastman Kodak Company, Apparatus and Optical Division, Rochester, N. Y. 14650.

Why even those who think they understand the situation in the infrared beyond photography may not

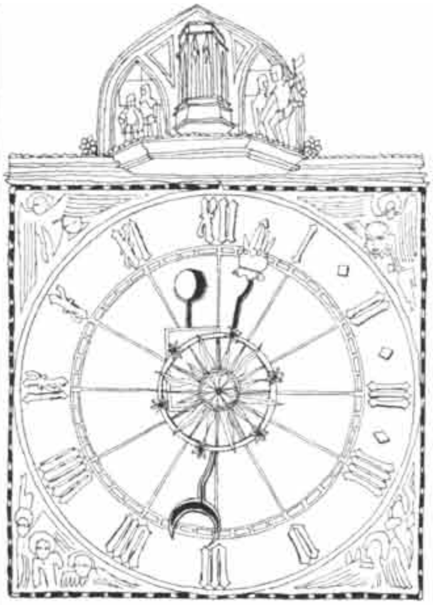
Most IRTRAN material and most work of infrared opticists wind up in military hands. Military considerations ultimately and inevitably have emotional content. Emotion suffuses the idea that someone who means you harm has technical devices for extending the senses that he and you both received at birth. You hope you know more about these means than he does and more about what to look for with these means. To feel this way goes along with the human condition. It sets one limit on the decisions of the declassifying boards.

The opposite limit is set by another realization which the members of these boards share with you. They too know that secrets stale quickly and that scientists in a position to freshen them up were born with a "need to know" but can produce no documents to prove it and often don't even want to. Intelligent military seek an optimum between these limits.

Two unclassified symposia on "Remote Sensing of Environment" and one classified one at the Institute of Science and Technology of the University of Michigan have given the military the thoughts of such diverse types as geologists, botanists, astronomers, conservationists, meteorologists, oceanographers, foresters, archaeologists, and half a dozen species of engineers. Everybody present knew that the already known applications will take care of themselves but wondered about the unknown ones that are in danger of dying unborn for want of cross-fertilization.

The final report of the symposia, identified as Document 4864-6-F, carries the following statement: "Qualified requesters may obtain copies of this document from: Defense Documentation Center, Cameron Station, Alexandria, Virginia." We have inquired how one qualifies as a requester and have been informed by the Defense Documentation Center that all reports it releases must be for use on a current Department of Defense contract or project.

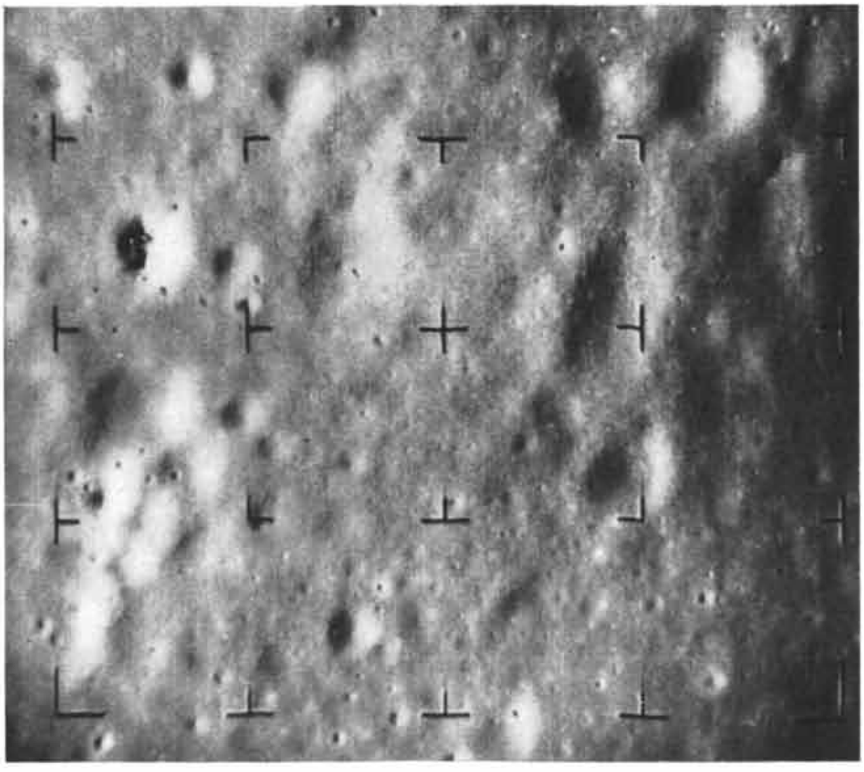
SCIENCE AND



Moon Microscopy

The 4,316 extraordinary photographs made by the space vehicle *Ranger 7* first and foremost indicate that the surface of the moon is not a loose quicksand hazardous to landing vehicles but is basically firm. That is the opinion of the investigators most closely connected with the Ranger project, notably Gerard P. Kuiper of the University of Arizona and Eugene M. Shoemaker of the U.S. Geological Survey. According to Kuiper, there may be a shallow top layer that "would be like crunchy snow." Shoemaker, noting that a few investigators see evidence in the photographs that a deep layer of dust covers the moon, said the pictures "proved to everybody that their preconceptions were right." He added that "there is no way we can measure directly the bearing strength... without putting something down on the lunar surface and patting it."

The evidence mainly in dispute is a rounding of small craters that was revealed by the photographs [*see illustration below*]. Shoemaker said the most decisive evidence of a firm surface is the shape of the primary craters: those caused by the high-speed impact of objects from space. If the surface were dust, he said, primary craters would have the appearance of punched holes; instead they have the shape that is characteristic of a dense medium. As



MOON AT THREE MILES appears in *Ranger 7* photograph. Rounding of crater edges, possibly through erosion, is a phenomenon revealed for the first time by *Ranger*. Dark crater at top left, about 300 feet in diameter, contains rock mass that apparently caused crater.

for the rounding, Shoemaker pointed out that secondary craters—those caused by the low-speed impact of material thrown out from large primary craters —"are rounded to start with." He added that both primary and secondary craters may also have been rounded by erosion of some kind, perhaps bombardment by micrometeorites.

Ranger's target area was the Sea of Clouds, one of the moon's maria, or dry seas. Officials of the U.S. space program chose the area because they believed the maria, which appear relatively smooth in contrast with the cragginess of other lunar areas, to be the most promising sites for the landing of manned spacecraft. The *Ranger* photographs bore out the impression of relative smoothness; Kuiper said the "chances of landing in a reasonably level area are . . . of the order of 99 percent." He also pointed out, however, that the photographs reveal some areas that "one should avoid as poison" in a manned landing because they have a "truly enormous" number of secondary craters.

Ranger began making photographs at a distance of about 1,000 miles from the moon. It made its last photograph at a distance of about 1,000 feet. Kuiper said the photographs mean that "the moon, which . . . in a good telescope can be brought to a distance of 500 miles equivalent, has been brought . . . to a distance of half a mile," making it possible to distinguish features as small as 1½ feet in diameter.

The U.S. program to put a manned Apollo spacecraft on the moon calls for two more Ranger flights. Late next year the first Surveyor, a vehicle that will land softly on the moon for a closer examination of the surface, is scheduled for launching. Next, starting in 1966, there will be a series of Lunar Orbiter flights, in which spacecraft will orbit the moon at low altitude, photographing large stretches of lunar terrain. The first landing of men is scheduled to take place by 1970.

Fall of Time Reversal?

A recent experiment in particle physics has thrown doubt on the crucial principle, built into almost every physical theory, that the laws of the universe remain unchanged when the direction of time is reversed. Known as time-reversal invariance, the principle states in effect that there is no way of telling if a motion picture film of an event (any event) is being run forward or backward. The ordinary clues that might reveal the direction are secondary, not fundamental.

The experiment that casts doubt on this principle was performed by James H. Christenson, James W. Cronin and Val L. Fitch of Princeton University and René Turlay, on leave from the Center for Nuclear Studies in France. Their work is reported in *Physical Review Letters.*

The Princeton group studied the mode of decay of the neutral K meson known as K_2^0, one of four members of the K family: K^-, K^+, K_1^0, K_2^0. The theory governing the decay of such particles includes the "*CPT*" rule, which states that particle reactions are indistinguishable from their time-reversed, antimatter mirror images. In this rule *C* stands for charge conjugation, which relates matter to antimatter; *P* stands for parity, which relates a system to its mirror image, and *T* stands for time reversal, which relates a system to one in which time flows in the opposite direction.

If *C* and *P* are conserved, the K_1^0 can decay into a pi-plus meson (π^+) and a pi-minus (π^-), but the K_2^0 is forbidden to do so. The reason is that the K_1^0 has a *CP* value of +1 (as determined by experiment) and the K_2^0 has a *CP* value of −1. It is also known that π^+ and π^- have a combined *CP* of +1. Therefore only the K_1^0 can decay into the two π's. If it were found that the K_2^0 also decayed into π^+ and π^-, it would mean that a state with a *CP* of −1 had changed into a state with a *CP* of +1. In this case the *CPT* rule could be preserved only if there had been a compensating violation of *T*, or time-reversal invariance.

Because it would provide a crucial test of the *CPT* rule, investigators had previously looked for the forbidden two-pi decay of the K_2^0, but they were unsuccessful. Previous experiments were sensitive enough to have detected one two-pi decay among 300 decays of other types. In the new group of experiments about one in every 500 K_2^0 decays was of the two-pi type; all told about 50 two-pi events were recorded.

The experiment will of course be repeated. In addition physicists will look for tests in which *T* alone can be tested apart from *CP*. One proposed test is to measure the spin, or angular momentum, of the mu meson when a K decays into a pi meson, a mu meson and a neutrino. The effect of time reversal is to change the sign of all momentum vectors and the spin vector. If time-reversal invariance is true, the number of mu mesons spinning in one direction with respect to a plane defined in a certain way should exactly equal the number spinning in the opposite direction. If the numbers are not equal, the fall of time-reversal invariance will be confirmed.

GUU Means Valine

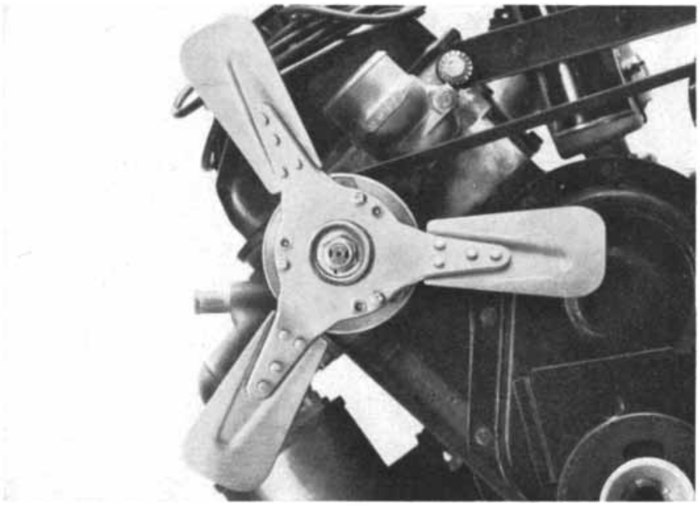
Three years ago at the Fifth International Congress of Biochemistry in Moscow the first "break" in the genetic code was described by Marshall W. Nirenberg of the National Institutes of Health. At the Sixth Congress, held in New York in July, Nirenberg described another major advance in the decipherment of the code: the effect of changing the sequence of code "letters" in one particular code group.

In the first round of code-breaking experiments Nirenberg and his associates, and independently Severo Ochoa and his associates at the New York University School of Medicine, worked out a genetic-code "dictionary" showing how code groups made up of the nucleic acid bases adenine (A), uracil (U), guanine (G) and cytosine (C) specified each of the 20 amino acids commonly found in protein molecules. Nirenberg's original experiment had shown that a synthetic molecule of ribonucleic acid (RNA) containing only U subunits led to the production of a synthetic protein chain consisting only of subunits of the amino acid phenylalanine. Thus phenylalanine was said to be specified by the code group UUU; it was assumed, but not proved, that the code groups were triplets, or groups of three bases each.

In the case of homogeneous triplets such as UUU there is no sequence problem. The early experiments, however, were unable to distinguish between various forms of mixed triplets such as UUG, UGU and GUU. Consequently it was found that some combination of two U's and one G coded for three different amino acids (leucine, valine and cysteine), but which sequence coded for which amino acid was unknown.

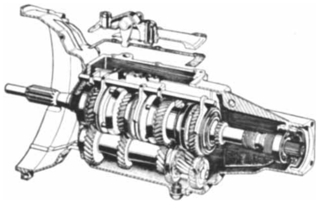
Nirenberg and his associate Philip Leder have now found that GUU is the code word for valine. This result was based on work with individual triplets —not chains—of GUU, UGU and UUG. The new work has shown that the code-reading mechanism can distinguish be-

The Peugeot's fan



doesn't function until engine temperature reaches 180°

Only one car in the world offers this feature. On other cars the fan runs all the time—even when the engine is cold. On the Peugeot 404, however, the fan runs only when it is needed—namely when engine temperature reaches 180°. As a result of the fan's freewheeling, you get 4 to 5 additional horsepower, faster warmup, reduced engine drag, lower noise level and, most important, substantial fuel savings.

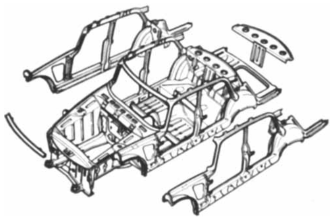


The Peugeot 404's transmission consists of four forward synchromesh gears and one reverse. All of the forward speeds are synchronized for smooth shifting. Because of the power and torque of the 404 engine, fourth speed is direct drive. The gears are of the noiseless helical type; ratios have been selected to match engine output for maximum performance and economy.

You'll notice that Peugeot's front windows don't have vents as most cars do. They're not needed, thanks to a unique fresh air ventilating system which

provides a smooth flow of air. Flush nozzles built into the dash distribute air wherever it is desired. Peugeot's ventilating system can even direct air at your feet for extra comfort. With the ventilating system on and the sunroof open, you can actually close all windows, which not only cuts down on street noises but gives you perfect ventilation without drafts.

The 404 is built with unit body construction. That is, the body is welded into one solid structure instead of being



bolted to a separate frame. One major advantage is the fact that greater strength is provided, with less weight. In addition, squeaks and rattles are far less likely to occur. The 404's body steel is constructed of heavy gauge steel (.9 mm thick) for durability and safety, consistent with the world-wide reputation of Peugeot.

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tween GUU and its mirror image UUG. Moreover, in the cases studied so far triplets are the shortest sequences that act as coding units. There had been some speculation that the cell might use a two-base code. Nirenberg's latest methods point the way to a new genetic-code dictionary in which each amino acid is associated with one or more specific triplets.

Pre-Columbian Solar Observatories

The Mississippi bottomland opposite St. Louis, notable for the largest Indian earthwork in the U.S., appears also to have been the site of four wooden constructions used to determine the sun's horizon position at various seasons of the year. Salvage archaeology during 1960 and 1961 in advance of highway construction among the mounds and plazas of the Cahokia site in Illinois uncovered a large number of four-foot-deep postholes averaging two feet in diameter. Plotting the positions of these postholes, Warren L. Wittry of the Cranbrook Institute of Science in Detroit discovered that they formed portions of four precise circles ranging from 240 to 480 feet in diameter. Wittry located 20 of an estimated 48 perimeter postholes in the second largest array, which has an associated carbon-14 date of about A.D. 1045. In 1963 a colleague found a 21st hole, this one near the circle's center but offset more than five feet to the east.

Tantalized by the knowledge that various Pueblo peoples of the Southwest kept track of the seasons by means of sun-sightings, Wittry wondered if lines of sight taken from the offset posthole would show any of the perimeter postholes to be in significant solstice or equinox positions. Taking into account the inclination of the ecliptic in A.D. 1000, he calculated the angle north of east for the summer solstice sunrise and found that one of the perimeter posthole positions was in line. Subsequent calculations have shown that the other three circles, although not yet known to possess central sighting points, also have postholes north and south of east potentially suited for a visual check of the summer and winter solstice sunrise.

Even quite crude observations of the means and extremes during the sun's annual tour of the horizon will produce a good approximation of the "tropical" (365.2422-day) year, the agriculturists' key calendar unit. For the builders of the Cahokia temple-mound complex, who lived for the most part on corn, beans and squash, an ability to forecast optimum planting times should have been ample reward for the trouble of constructing these sun-sighting "woodhenges."

Outside Educators

A student at a small college in the U.S. is likely to find his course of study proscribed by the fact that the teaching staff is unable to offer scientific instruction beyond the elementary level. To enlarge the educational opportunities of such a student the National Science Foundation seven years ago allocated funds to the American Institute of Physics and similar organizations in other fields to establish a Visiting Scientists Program. William W. Watson of Yale University, who has been dispatching physicists to small colleges under the auspices of the program, says the visitors have provided valuable stimulation in many instances but "the situation in physics teaching is still 'the rich get richer and the poor get poorer.' The large universities get the best people. The small, poor colleges get no one with advanced training."

This year an offshoot of the program will offer undergraduates at one small college more than stimulation. Graduate students from Northern universities will augment the faculty for an entire year at Tougaloo Southern Christian College near Jackson, Miss., to teach a broadened physics curriculum. The new courses were designed by John B. Garner, instructor in physics at the college, and David Finkelstein, professor of physics at the Belfer Graduate School of Science of Yeshiva University, who will spend a week lecturing at Tougaloo as part of the Visiting Scientists Program.

The additional courses call for two instructors with advanced training. The work of these two will be shared by 18 graduate students, each of whom will devote a month of the school year to teaching at Tougaloo. This scheme does not deprive a "donor" university of an indispensable faculty member, nor does it deprive a professor of his graduate-student aide for more than a month. Yet it will substantively enrich the instruction at one small and poor college and might prove applicable to others.

Vanishing Detergents

There should be less froth on America's lakes and rivers and in its wells and tap water next year: the "hard" detergents that are responsible for most of the foaming are to be withdrawn

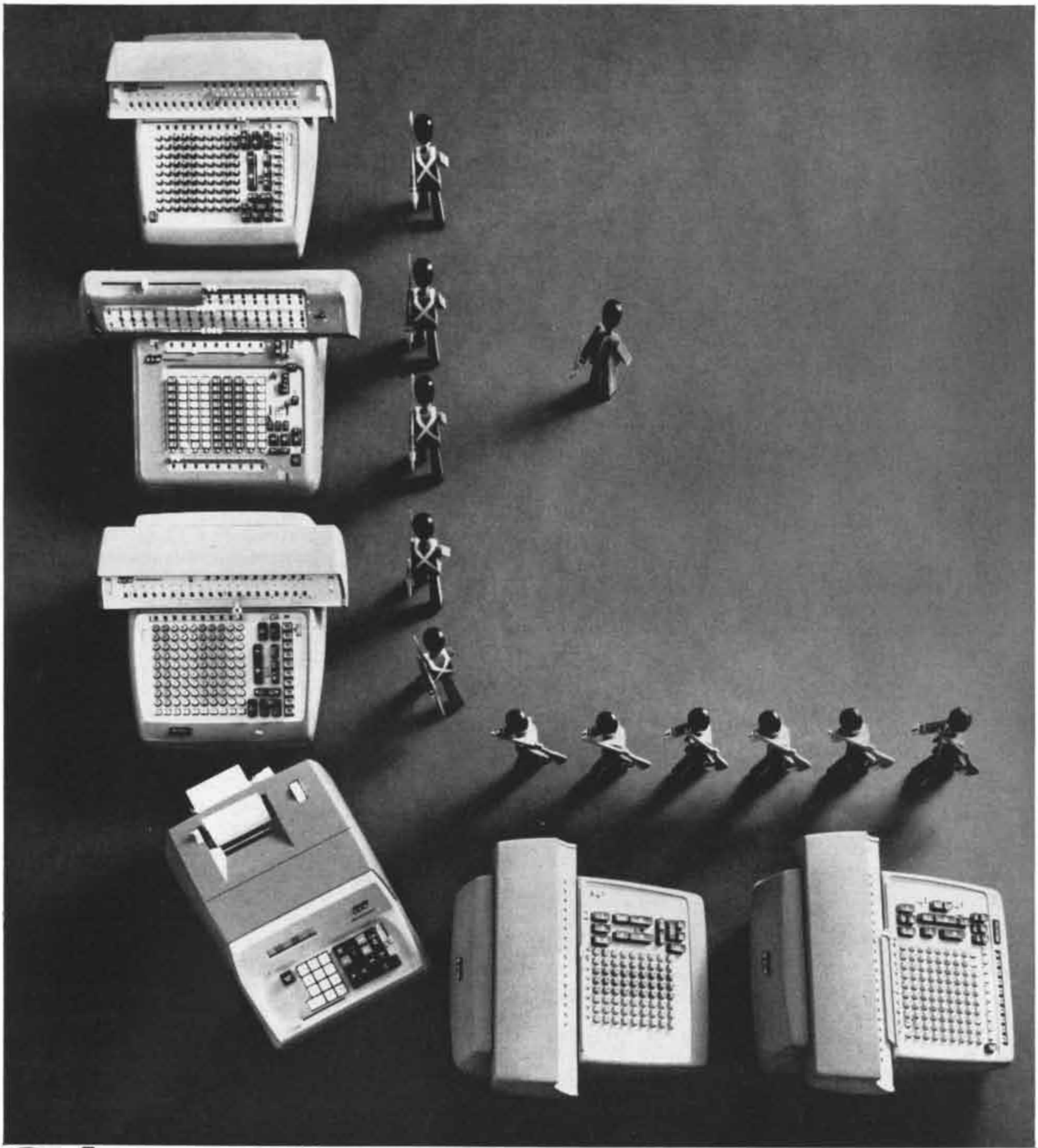
from the market by June 30, 1965. Although there is experimental evidence that detergent residues are not harmful to people or to aquatic animals and plants, the very appearance of the foams as reminders that much waste water is reused brought demands for Federal or state controls. Last year the major detergent manufacturers voluntarily agreed to switch to "soft" detergents. The original date for the changeover—the end of 1965—has now been advanced by six months.

The foaming problem arose largely because the most commonly used "surface-active" ingredient of the hard detergents, alkylbenzene sulfonate (ABS), cannot readily be degraded by bacteria in sewage-treatment plants or in nature. The component of ABS that resists bacterial attack is the branched-chain alkyl group. The new soft detergents have as their surface-active ingredient a slightly different compound called linear alkylate sulfonate (LAS), in which the alkyl group is strung out in a straight chain and is therefore subject to faster decomposition.

Slow Fade

Powerful X rays from the direction of the Crab nebula are *not* produced by a tiny, superdense "neutron star" at the center of the nebula. The neutron-star hypothesis had been formulated to account for the existence of two extremely intense sources of X radiation in the sky; the two sources, one in the vicinity of the constellation of Scorpius and the other coincident with the Crab nebula, were discovered recently by rocket-borne detection devices (see "X-ray Astronomy," by Herbert Friedman; Scientific American, June).

The experiment that disproved the existence of a neutron star at the center of the Crab nebula was conducted on July 5 by Friedman, Stuart Bowyer, Edward T. Byram and Talbot A. Chubb of the Naval Research Laboratory. On that day the Crab nebula was due to be "occulted," or blacked out, by the moon—a rare event that will not occur again until 1972. An Aerobee rocket bearing X-ray-sensitive detectors was fired from the White Sands Missile Range at precisely 3:42.5 P.M., just as the nebula began to pass behind the edge of the moon. If the neutron-star hypothesis were correct, the X-ray source would have disappeared abruptly within one or two seconds of arc. Instead the X-ray intensity dwindled during the five-minute occultation in proportion to the nebular surface blacked out by the



moon. The source appears to be an extended region about one light-year across near the center of the nebula. (The nebula itself is six light-years in diameter.) Crude spectral measurements indicate that the temperature in this region may be higher than 10 million degrees Kelvin.

Although the neutron-star hypothesis has proved incorrect in the case of the Crab nebula, Friedman and his colleagues still hope to detect a neutron star elsewhere in the sky, perhaps in the direction of the extremely powerful X-ray source in Scorpius.

New Superconductors

Only a few metals and alloys exhibit the property of superconductivity: the complete disappearance of electrical resistance at temperatures close to absolute zero. Now two metals that are not superconducting either alone or when alloyed in a normal manner have been made superconducting by ultrarapid cooling. Pol Duwez, Ronald H. Willens and Huey-Lin Luo of the California Institute of Technology created the new alloy by cooling a molten mixture of gold and germanium at a rate of more than four million degrees Fahrenheit per second.

They do this by allowing a droplet of the melt to fall from a crucible past a photoelectric cell that triggers a pneumatic hammer. The hammer slams the droplet against a copper anvil, spreading it almost instantaneously over the copper surface, which is a good absorber of heat. The droplet is cooled from more than 3,500 degrees F. to room temperature in less than a thousandth of a second, and the resulting thin foil is superconducting.

What happens is that the quick-freezing process prevents the formation of a normal crystal structure in the alloy. If a melt of the two metals were allowed to cool gradually, the resulting alloy would have a regular pattern of discrete crystals of gold and germanium and would not be superconducting. Sudden cooling, however, freezes the well-mixed gold and germanium atoms into a new pattern, with new chemical bonding properties, much finer grain structure and crystals so small that they can be discerned only with an electron microscope. The main significance of the work, according to Duwez, lies not in the creation of a new superconducting alloy but in the discovery that extremely rapid cooling from the liquid state can change the structure and characteristics of metals to this extent.

Basic Research at Honeywell
Research Center
Hopkins, Minnesota

Beam Surface Interactions in Ultra High Vacuum Systems

In high vacuums the quantity of gases adsorbed on the surfaces in the vacuum is much greater than that contained in the volume. New studies of the interactions of beams with surfaces are making possible further understanding of the reactions that occur at the interface of a solid and a gas in a high vacuum environment.

Within an ultra high vacuum system, the reactions that take place at the surface may have an appreciable effect on the surrounding vacuum and on the properties of the solid itself. Continued progress in the fields of semiconductors, thin films, the vacuum phenomena related to space exploration and others have pressed the state of the art of vacuum physics. This field has long been of interest but many tools were lacking. The explosion in vacuum technology since the early 1950's, however, has resulted in considerable basic progress.

Recently lower pressures have been achieved and gauges and other instrumentation improved. We are now able to probe some fundamental questions:

1. What is the surface of a substrate like? What are the surface atom layers like? Is there an oxidized layer?
2. What is sitting down or adsorbed on the surface? With what energy is it bound to the surface?
3. Beams of photons, electrons, ions or atoms may strike the surface. When they do, what happens? What are the interactions?

If we could answer all these questions we could develop a model with specific constants.

Honeywell scientists have chosen a research technique whereby particle beams are used to probe a surface in an ultra high vacuum environment. Components leaving the surface (that is, neutral atoms, neutral molecules and positive or negative ions) are analyzed with a mass spectrometer.

A series of studies is being made to see what effect varying parameters have on the components leaving the surface. The kind of beam used, the kind of substrate and the temperature of the substrate as well as the content of the vacuum environment are varied.

Honeywell scientists use an evacuated system pumped by cryogenic and ion means resulting in a background pressure of about 10^{-10} Torr.

A target with surface temperatures controlled by heaters is mounted in an interaction chamber and bombarded. All particles leaving the target pass through a mass spectrometer analyzer. Probing is done with very low density beams since the detection system permits recording of single ions or partial pressures as low as 10^{-16} Torr.

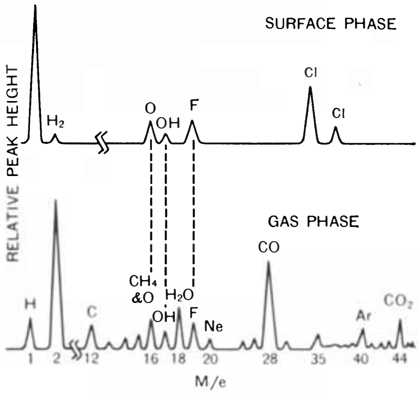


FIG. 1 MASS SPECTRUM

Lower Plot—gas phase components in the system at a total pressure of about 1×10^{-9} Torr.

Upper Plot—spectrum of surface phase obtained by bombarding single crystal nickel with electrons and analyzing the desorbed ion components.

These experiments have produced several unexpected observations:

1. Electron bombardment will desorb neutral molecules, suggesting an interaction between the electron and the adsorbed molecule.
2. Ions are desorbed at the same time and appear to be fragments of the parent adsorbed molecule, suggesting ion fragment desorption.
3. No parent molecule ion desorption was observed.

These studies indicate the existence of a whole spectrum of electron-induced ion desorbed species, permitting the analysis of surface phase components in a manner similar to that used for gas phase components. The technique also permits continuous observation of surface components as parameters are varied.

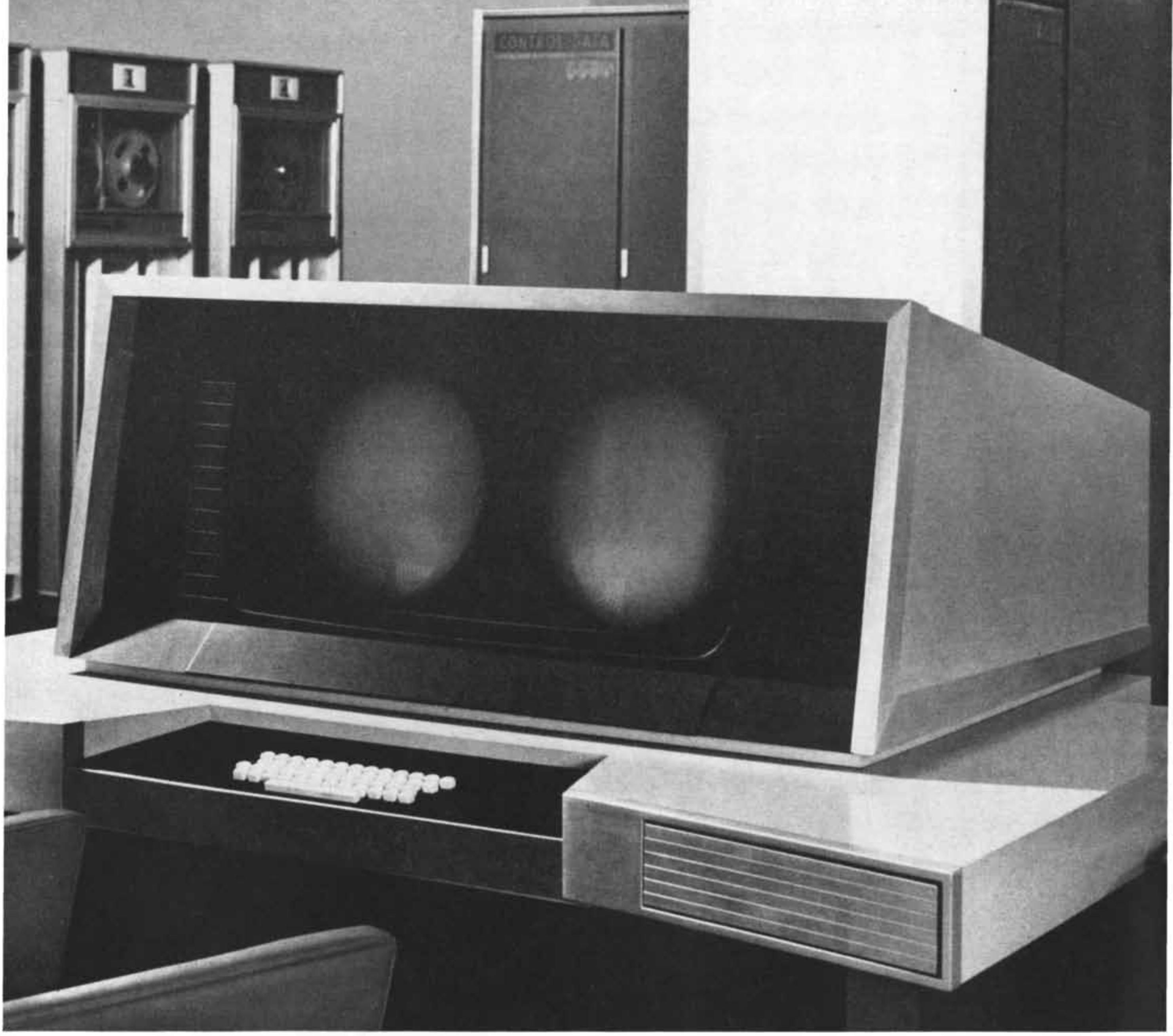
Bombardment with U.V. photons has also desorbed neutral molecules. This suggests that photon interaction might be used to clean surfaces in a vacuum without any heating effects.

Work is continuing at Honeywell's Research Center and as more parameters are introduced even more understanding seems possible. As an example, a ruby laser was used to bombard the target causing thermal desorption and permanent degassing of an extremely small area of the target. This technique will permit further exploration and comparisons of the degassed spot and surrounding surfaces.

Although a long way from a final theory, the new techniques already have provided information of value in programs as diverse as electrical contacts, U.V. detectors and space instrumentation.

If you are engaged in vacuum surface physics and wish to know more about Honeywell's work in this area you are invited to write Mr. David Lichtman, Honeywell Research Center, Hopkins, Minnesota. If you are interested in a career at Honeywell's Research Center and hold an advanced degree, write Dr. John Dempsey, Director of Research at this same address.

Honeywell



on half-a-million Fairchild transistors.

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PROBABILITY

The real world confronts the mathematician with events that are not strictly predictable. The methods he has developed to deal with such events have opened new domains of pure mathematics

by Mark Kac

A secretary has typed 10 letters and addressed 10 envelopes. If she now puts the letters in the envelopes entirely at random (without looking at the addresses), what is the probability that not a single letter will wind up in its correct envelope? It may surprise the reader to learn that the probability is better than one chance in three: more specifically, it is almost 1/2.71828.... (This famous number 2.71828..., or e, the base of the natural logarithms, turns out to be an important one in the theory of probability and comes up again and again, as we shall see.)

The method used to solve the problem is called combinatorial analysis. An older and more familiar example of problems in combinatorial analysis is: What is the probability of drawing a flush in a single deal of five cards from a deck of 52? Combinatorial analysis has more profound and more practical applications, of course, than estimating the chances of poker hands or answering amusing questions about the hypothetical behavior of absentminded secretaries. It has become an extremely useful branch of mathematics. But its principles are best illustrated by simple examples. Let us work out the poker problem in detail so that we can perceive some of its probabilistic implications.

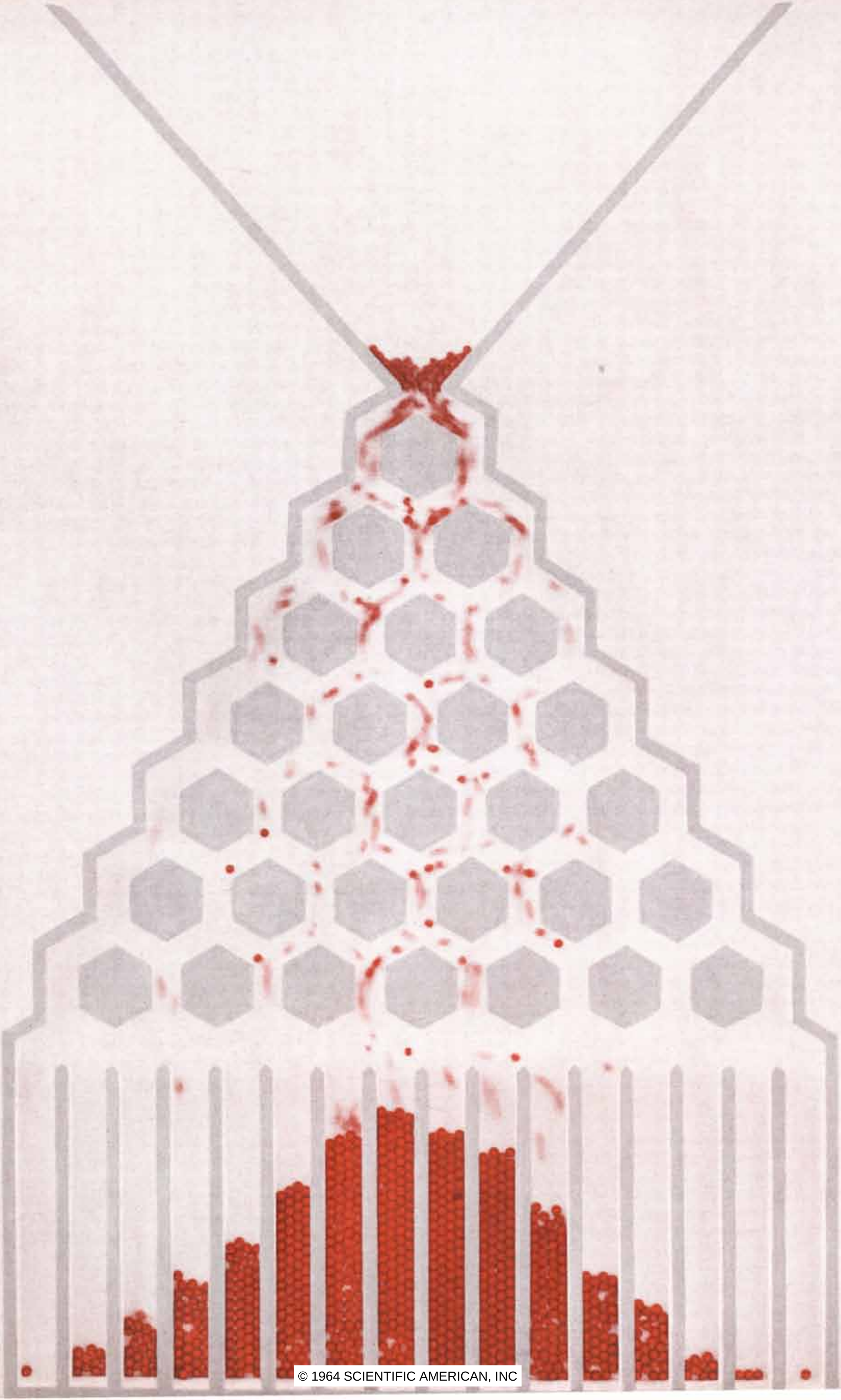
Pierre Simon de Laplace (1749–1827) based an entire theory of probability on combinatorial analysis by defining probability as $p = n/N$. This expression states that the probability of an event is the ratio of the number of ways in which the event can be realized (n) to the total number of possible events (N), provided that all the possible events are equally likely—an important proviso. The probability of a poker flush therefore is the ratio of the number of possible flushes to the total number of possible poker hands. The problem of combinatorial analysis is to calculate both numbers.

Let us start with a simpler case involving more manageable numbers. Given a set of four objects, *A, B, C, D,* how many subsets, or combinations, of two objects can be made from them? It is easy to answer by simple pairing and counting: there are six possible combinations of size 2, *AB, AC, AD, BC, BD, CD.* As we go on to larger numbers of objects, however, this process soon becomes all but impossible. We must find shortcuts—ways to make the calculations without actually counting. (Combinatorial analysis is sometimes called "counting without counting.")

Suppose we add a fifth object and consider how many pairs can be formed from the five. It is apparent that the new object, *E*, adds just four to the total of possible pairs, because it can combine with each of the other four. So the total is 6 + 4, or 10, possible pairs. To put it in the conventional symbols of combinatorial analysis, we have $C(5,2) = C(4,2) + C(4,1)$. C represents the number of combinations, and the numbers in parentheses stand respectively for the total number of objects and the number in each subset: for instance, $C(5,2)$ means the combinations of five objects taken two at a time. To calculate on the same principle the number of combinations of four objects that could be made out of a total of 10 objects, we could write $C(10,4) = C(9,4) + C(9,3)$ and then continue the reduction to smaller and smaller numbers until we finally computed the answer by simple addition of all the numbers. In practice what we actually do in such a case is to build the C's from the ground up (the bookkeeping is easier).

The whole scheme is conveniently summarized in a handy table known as Pascal's triangle after Blaise Pascal (1623–1662), one of the founders of the theory of probability. The triangle is made up of the coefficients of the binomial expansion, each successive row representing the next higher power [*see top illustration on page 94*]. Each number in the table is the sum of the two numbers to the right and the left of it in the row above. The number of combinations for any set of objects can be read from left to right across a row. For example, the fourth row describes the possible combinations when the total number of objects is four: reading from the left, we have first the number 1, for the "empty set" (con-

PROBABILITY DEMONSTRATOR shown on the opposite page mechanically produces an approximation to the bell-shaped "normal," or Gaussian, distribution. The little red balls rolling from the reservoir at the top pass an array of hexagonal obstacles and collect in receptacles at the bottom. At each obstacle the probability ought in theory to be one-half that a ball will go to the right and one-half that it will go to the left. Thus the balls tend to distribute themselves according to the proportions of Pascal's triangle, shown in the top illustration on page 94. In the photograph the balls falling through the array of obstacles are blurred because of their motion. The balls have not produced the full distribution curve because some are still moving through the channels. The apparatus is known as a Galton Board after Sir Francis Galton, who constructed the first one. The version shown here is based on a design by the Science Materials Center, patented under the name "Hexstat."



1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

PASCAL'S TRIANGLE, an aid to calculating probabilities, is made up of the coefficients of the binomial expansion. Each number is the sum of the two numbers immediately above it. Some of its characteristics and applications are discussed in the text.

taining no objects); then 4, the number of subsets containing one object; then 6, the number of possible combinations of two objects; then 4, the number of three-object combinations, and finally 1 for the "full set" of four objects. With this table, to find the number of quadruplets that can be formed from a total of 10 objects one goes to the 10th row, reads across five steps to the right and finds the answer: 210.

Even the Pascal triangle becomes inconvenient when it has to be extended to large numbers such as are involved in our poker problem. Fortunately the pioneers of the theory of probability were able to work out and prove a simple general formula. This now familiar formula, in which (n,r) means n objects taken r at a time and "!" is the symbol meaning "factorial," is

$$C(n,r) = \frac{n!}{r!(n-r)!}.$$

In the case of $C(10,4)$ the formula—simplified by dividing both numerator and denominator by $(n-r)!$—becomes

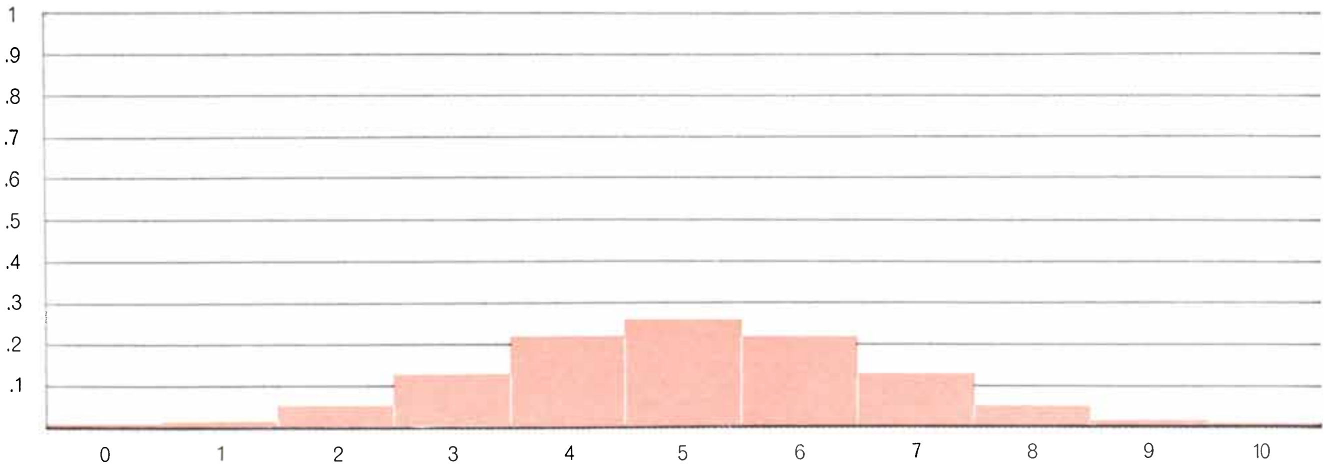
$$C(10,4) = \frac{10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} = 210.$$

Now it is not difficult to compute the probability of a poker flush. There are $C(13,5)$ possible flushes in each suit, a total of $4C(13,5)$ in the four suits. The total number of possible poker hands is $C(52,5)$. Hence the probability of getting a flush out of all the hands that might be dealt is

$$\frac{4C(13,5)}{C(52,5)} = \frac{4\,\dfrac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}}{\dfrac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}} = \frac{33}{16{,}600}.$$

This comes out to about two chances in 1,000 of drawing a flush of any kind in a five-card deal from a full deck.

Let us proceed to investigate probability further by the classic device of coin-tossing. Suppose I toss a coin 10 times; what is the probability that in the 10 throws I shall get exactly four



PROBABILITY OF HEADS in 10 tosses of a coin produces a histogram that is reminiscent of the normal distribution. As the 10th row of Pascal's triangle shows, there are 210 possible sequences containing exactly four heads out of a total of 1,024 possible sequences of heads in 10 tosses. Thus the chance of four heads is about 21 percent. Roughly the chances of zero, one, two, three, four and five heads in 10 tosses (*horizontal scale*) are respectively .001, .01, .045, .12, .21 and .25 (*vertical scale*). The probabilities and the bars representing them occur in descending order for six through 10 heads in 10 tosses. On the same scale a histogram for 10,000 tosses would be much wider and lower, and would have to be rescaled to bring out its relation to the normal curve.

heads? Looking at the 10th row of the Pascal triangle, we see that the possible sequences of heads and/or tails for 10 tosses add up to a total of 1,024. In this total there are 210 sequences containing exactly four heads. Therefore, if the coin-tossing is "honest," in the sense that all the 1,024 possible outcomes are equally likely, the probability of just four heads in 10 throws is 210/1,024, or roughly 21 percent.

The sum of all the entries in any given row (numbered n) of the Pascal triangle is equal to 2 to the nth power (for example, $1{,}024 = 2^{10}$). Thus in general the probability of tossing exactly k heads in a sequence of n throws is $C(n,k)/2^n$. Suppose we plot the various probabilities of tossing exactly 0, 1, 2, 3 and so on up to 10 heads in 10 throws in the form of a series of rectangles, with the height of each rectangle representing the probability [*see bottom illustration on opposite page*]. The graph peaks at the center (a probability of 252/1,024 for five heads) and tapers off gradually to both sides (down to probabilities of 1/1,024 for no heads and for 10 heads). If we plot the same kind of graph for 10,000 tosses, it becomes much wider and lower: the high point (for 5,000 heads) is not in the neighborhood of 25 percent but only $1/100\sqrt{\pi}$, or approximately .56 percent. (It may seem odd that in increasing the number of tosses we greatly reduce the chances of heads coming up exactly half the time, but the oddity disappears as soon as one realizes that a strict 50–50 division between heads and tails is still only one of the possible outcomes, and with each toss we have increased the total number of possible results.)

Drawn on the basis I have just described, the probability graph for a large number of tosses is so flat that it is hardly distinguishable from a straight line. But by increasing the heights of all the rectangles by a certain factor ($\sqrt{n/2}$) and shrinking the width of the base by the same factor, one can see that the tops of the rectangles trace out a symmetrical curve with the peak in the middle. The larger the number of tosses, the closer this profile comes to a smooth, continuous curve, which is described by the equation

$$y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

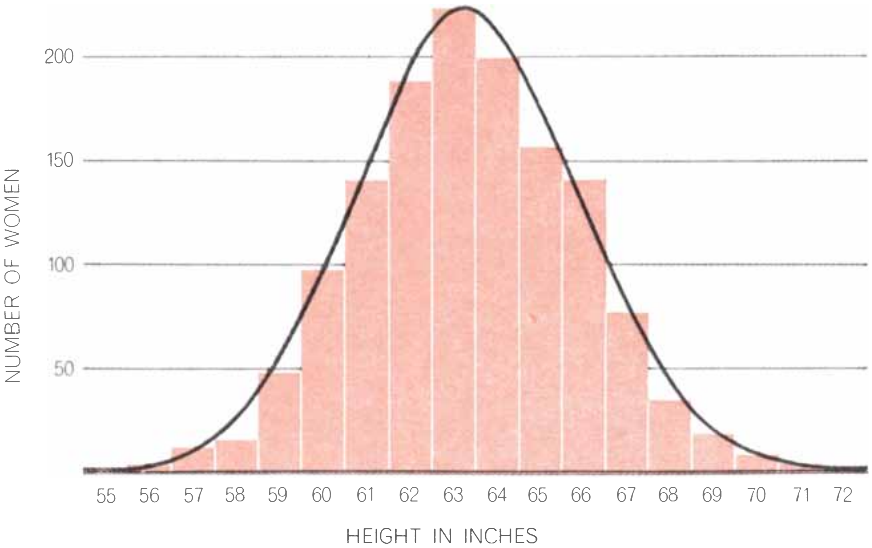
The e is our celebrated number 2.71828..., the base of the natural logarithms. (If a bank were foolish enough to offer interest at the annual rate of 100 percent and were to compound this interest continuously—not just daily, hourly or even every second but every instant—one dollar would grow to $2.71828... at the end of a year.)

The close approach of the probability diagram to a continuous curve with many tosses of a coin illustrates what is called a law of large numbers. If an "honest" coin is tossed hundreds of thousands or millions of times, the distribution of heads in the series of trials, when properly centered and scaled on a graph, will follow almost exactly the curve whose formula I have just given. This curve has become one of the most celebrated in science. Known as the "normal" or "Gaussian" curve, it has been used (with varying degrees of justification) to describe the distribution of heights of men and of women, the sizes of peas, the weights of newborn babies, the velocities of particles in a gas and numerous other properties of the physical and biological worlds.

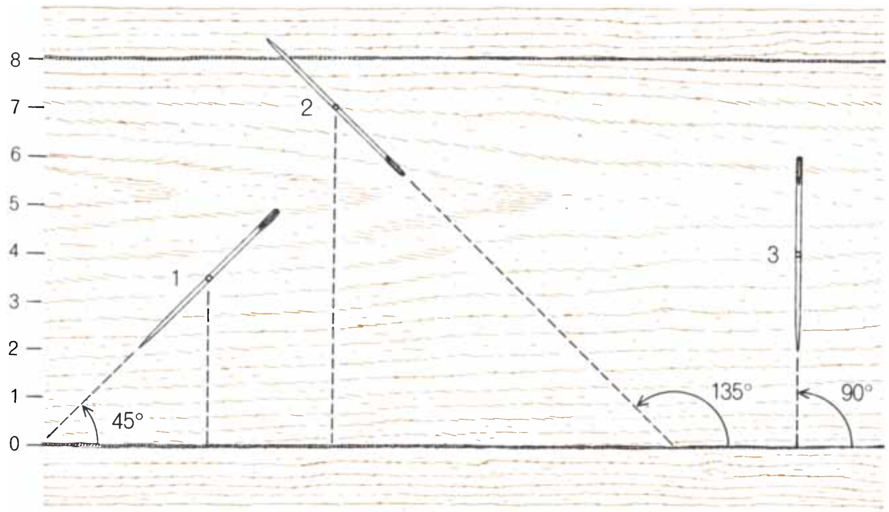
The remarkable connection between coin-tossing and the normal curve was both gratifying and suggestive. It provided one of the main stimuli for the further development of probability theory. It also formed the basis for the "random walk" model of tracing the paths of particles. This in turn solved the mystery of Brownian motion, thus establishing the foundations of modern atomic theory.

Probability today is a cornerstone of all the sciences, and its daughter, the science of statistics, enters into all human activities. How prophetic, in retrospect, are the words of Laplace in his pioneering work *Théorie analytique des probabilités*, published in 1812: "It is remarkable that a science which began with the consideration of games of chance should have become the most important object of human knowledge. ... The most important questions of life are, for the most part, really only problems of probability."

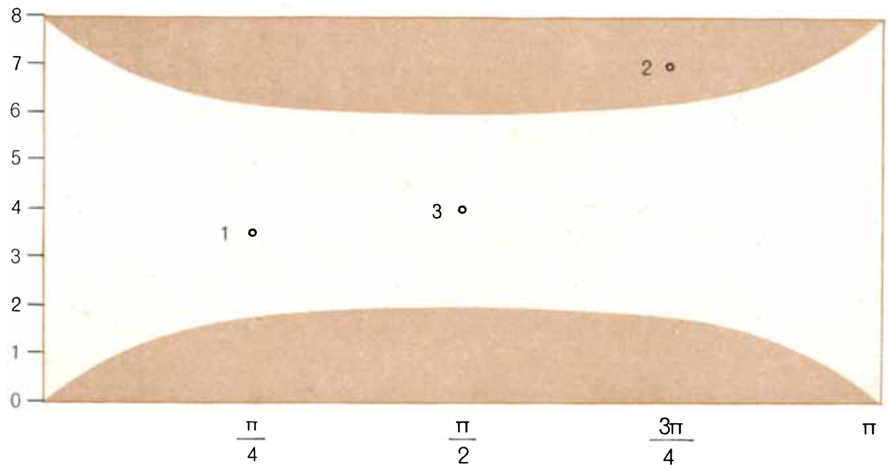
It seems to be a characteristic of "the most important objects of human knowledge" that they generally take a long time to become established as such. After Laplace interest in probability theory declined, and through the rest of the 19th century and the first two decades of the 20th it all but disappeared as a mathematical discipline. Only a few mathematicians went on with the work; among these were the brilliant and original Russian mathematicians P. L. Chebyshëv and his pupil A. A. Markov (which accounts for the strong development of probability theory in the U.S.S.R. today). There were spectacular applications of probability theory to physics, not only by Albert Einstein and Marian Smoluchowski in their solution of the problem of Brownian motion but also by James Clerk Maxwell, Ludwig Boltzmann and Josiah Willard Gibbs in the kinetic theory of gases. At the turn of the century Henri Poincaré and David Hilbert,



HEIGHTS OF WOMEN produce a histogram to which the normal-distribution curve can be fitted. There were 1,375 women in this sample population. The bell-shaped curve conforms to many other empirical distributions found in the physical and biological worlds.



BUFFON NEEDLE PROBLEM involves the probability that a needle shorter than the width of a plank will fall across the crack between two planks. Here each needle is half as long as the plank is wide. The eight units used to measure the plank represent inches.



ABSTRACT DIAGRAM also shows positions of the three needles. The horizontal scale represents the angle of each needle with respect to the bottom edge of the plank. The angle is given in terms of π, which is defined as 180 degrees. The vertical scale is the width of the plank in inches. The three dots are the center points of the needles. Called the "sample space," the rectangle represents all the possible positions in which a needle can fall. The dark colored areas cover all the positions in which a needle lies across a crack.

the two greatest mathematicians of the day, tried to revive interest in probability theory, but in spite of their original and provocative contributions there was remarkably little response.

Why this apathy toward the subject among professional mathematicians? There were various reasons. The main one was the feeling that the entire theory seemed to be built on loose and nonrigorous foundations. Laplace's definition of probability, for instance, is based on the assumption that all the possible outcomes in question are equally likely; since this notion itself is a statement of probability, the definition appears to be a circular one. And that was not the worst objection. The field was plagued with apparent paradoxes and other difficulties. The rising standards of rigor in all branches of mathematics made probability seem an unprofitable subject to cultivate.

In the 1930's, however, it was restored to high standing among mathematicians by a significant clarification of its basic concepts and by its relation to measure theory, a branch of mathematics that goes back to Euclid and that early in this century was greatly extended and generalized by the French mathematicians Émile Borel and Henri Lebesgue. To understand and appreciate this development let us start with a celebrated problem in geometrical probability known as the Buffon needle problem. If a needle of a certain length (say four inches) is thrown at random on a floor made of planks wider than that length (say eight inches wide), what is the probability that the needle will fall across a crack between two planks? We can define the position of the needle at each throw by noting the location of the midpoint of the needle on a plank and the angle between the needle and a given crack [*see upper illustration at left*]. Now, we can also show the various possible positions of the needle by means of an abstract diagram in the form of a rectangle [*see lower illustration at left*], in which the height represents the width of the plank and the base represents the angle (in terms of π, with π equal to 180 degrees, $\pi/2$ equal to 90 degrees and so on).

This rectangle as a whole, whose area is πd, represents all the possible positions in which the needle can fall. Technically it is called the "sample space," a general term used to denote all the possible outcomes in any probability experiment. (In tossing 10 coins the sample space is the set of all the 1,024 possible 10-item sequences of heads and/or tails.) In the needle experiment what part of the area of the rectangle corresponds to those positions of the needle in which it crosses a crack? This can be calculated by simple trigonometry, and it is represented by two sections within the rectangle with curved boundaries. Their combined area, which can be calculated by elementary calculus, turns out to be 2 times l, the length of the needle.

Now, if all the possible positions of the needle are equally likely, then the probability of the needle falling on a crack is the ratio of the dark colored areas in the illustration to the total area of the rectangle, or $2l/\pi d$. This is where the theory stumbles over its own arbitrary assumption. There is really no compelling reason to treat all the points in this abstract rectangle as equally likely, but the assumption is so natural as to appear inevitable. The degree of arbitrariness was dramatized by the French mathematician J. L. F. Bertrand, who devised examples (known as Bertrand paradoxes) in which, from assumptions that seemed equally natural, he could obtain quite different answers to a probability problem.

This was an unhappy situation, and a deeper understanding of the role and of the nature of probabilistic assumptions

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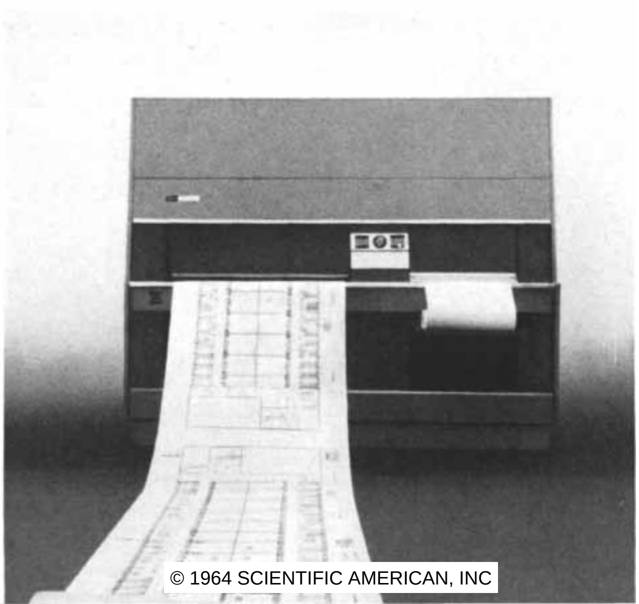

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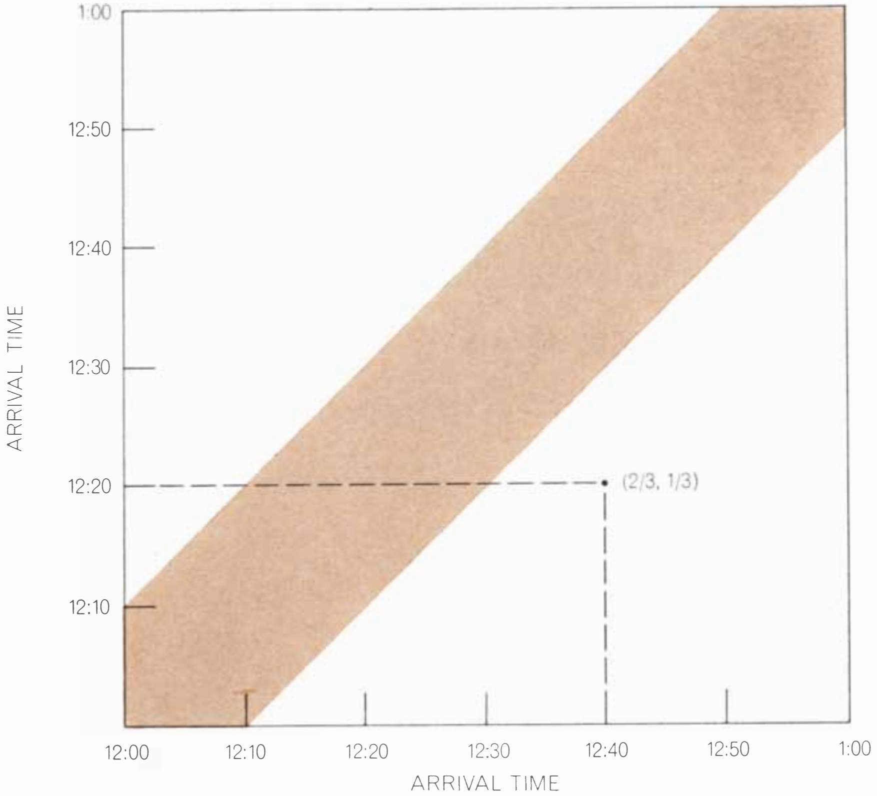
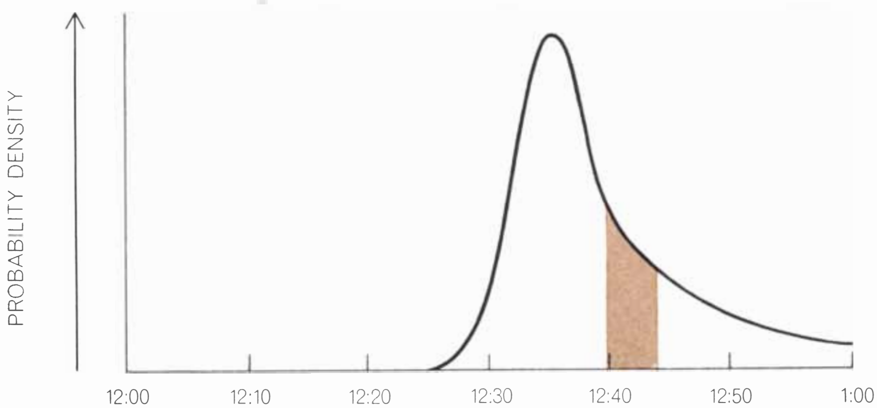
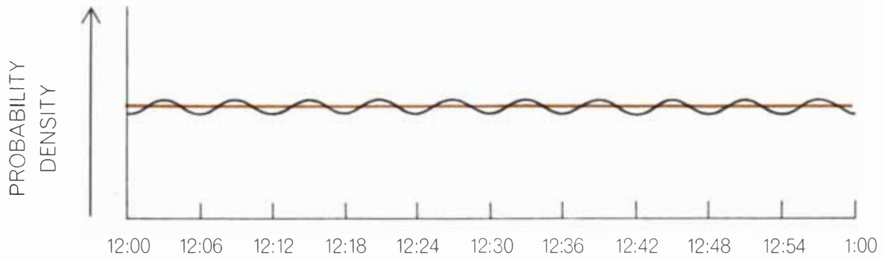


POSSIBLE ARRIVAL TIMES of two suburbanites planning to meet at a library between 12:00 noon and 1:00 P.M. can be plotted. Arrival times for one person are on the vertical scale, for the other on the horizontal scale. Colored area covers region corresponding to a meeting. In order to meet they must arrive at library within 10 minutes of each other. As can be seen, if one arrives at 12:20 (a third of the way up) and the other at 12:40 (two-thirds of the way across), they will not meet. The (1/3, 2/3) point falls outside the shaded region.



PROBABILITY-DENSITY CURVE illustrates degree of unpredictability of arrival time if there is only one train, coming in at 12:20. Most likely meeting time at library is 12:35. Area of shaded portion represents probability of arrival between 12:40 and 12:44 P.M.



TRAINS ARRIVING EVERY SIX MINUTES give density curve shown by black line. The colored straight line is the "curve" in case all the arrival times are equally likely.

was called for. I can best explain the modern view of these matters by means of another example.

Suppose two friends living in different suburbs of New York City want to meet in front of the Forty-second Street Public Library at noontime. Railroad schedules (and performance) being what they are, the friends can only count on arriving sometime between 12:00 noon and 1:00 P.M. They agree to show up at the library somewhere in that interval, with the stipulation that, in order not to waste too much time waiting, each will wait only 10 minutes after arriving and then leave if the other has not shown up. What is the probability of their actually meeting?

I should mention that, although this case is admittedly artificial, it is by no means a trivial problem. Extended to many members instead of just two, it is analogous to (but far simpler than) an important unsolved problem in statistical mechanics whose solution would shed much light on the theory of changes of states of matter—for instance, from solid to liquid.

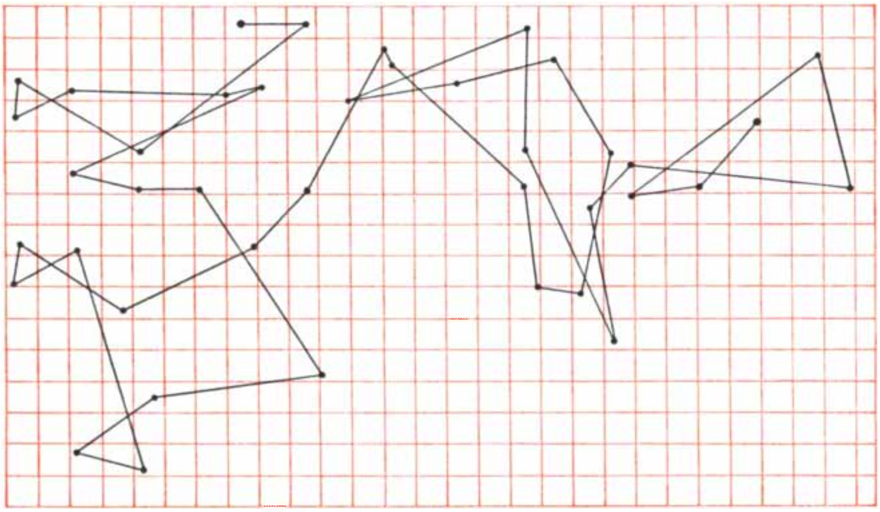
If we assume that each of the two friends may arrive anytime between 12:00 and 1:00, we can plot a geometrical "sample space" as in the Buffon needle problem. One person's possible times of arrival are denoted on the x axis, the other's on the y axis [*see top illustration on this page*]. We can then designate every possible pair of arrival times by a point in the square graph. Those points that lie within the part of the square that represents arrival times no more than 10 minutes apart will signify a meeting; all the other points will mean "no meeting." Taking the ratio of the two areas as the probability, as in the needle problem, we can calculate that the probability of the two friends meeting is 11/36—not quite one chance in three.

This case makes clear that we have made two different assumptions. Let us analyze them in a more general context.

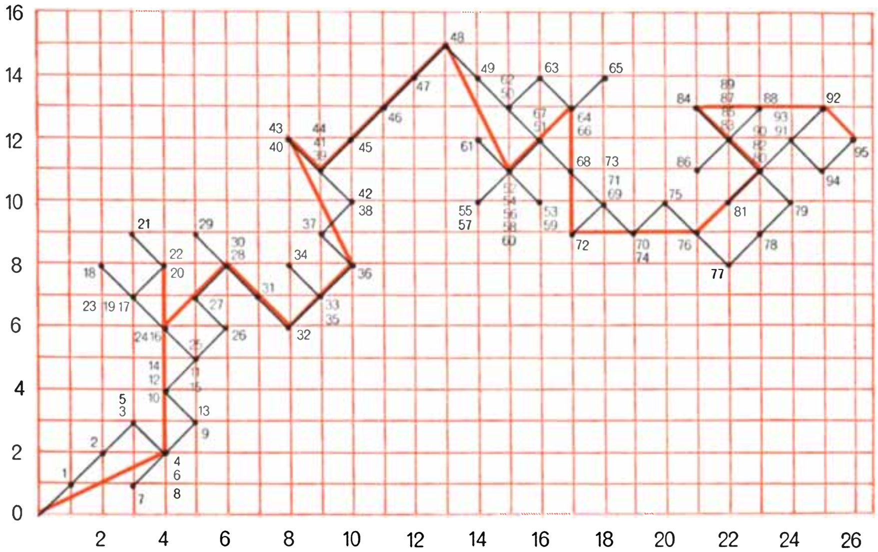
In very general terms probability theory, as a mathematical discipline, is concerned with the problem of calculating the probabilities of complex events consisting of collections of "elementary" events whose probabilities are known or postulated. For example, in rolling two dice the appearance of a 10 is a "complex" event that consists of three elementary events: (1) the first die shows a 4 and the second a 6, (2) the first die shows a 5 and the second a 5 and

(3) the first die shows a 6 and the second a 4. The meeting of our two friends is also a complex event; an example of an elementary event would be the arrival of one of the friends in the interval between 12:20 and 12:25.

In our calculation of the probability of the two friends meeting, the first assumption we made was that each of the two individuals may arrive anytime between 12:00 and 1:00, all times of arrival being "equally likely." (The corresponding assumption about the dice is that any one of the six faces of each die may come up with equal probability.) But if each person is limited to only one train scheduled to arrive at Grand Central during the hour (say at 12:20 or later), this assumption is completely unrealistic: he will certainly not arrive in the early part of the hour. The situation corresponds to the two dice being loaded. On the other hand, if there are six scheduled trains, due to arrive at 10-minute intervals from 12:00 on, and if they tend to be haphazardly off schedule, the assumption becomes more reasonable, although it may still not be strictly correct to say that all the times of arrival are equally probable.



BROWNIAN PATH is taken by a particle being "kicked around" by molecules of a surrounding liquid or gas. A stochastic process (it varies continuously with time), Brownian motion can be analyzed and modeled (*illustration below*) by probabilistic techniques.



BROWNIAN MODEL can be constructed from records of coin tosses. Two series of 90-odd tosses of a coin were used, one plotted on the horizontal scale, the other on the vertical scale. The cumulative total of tails at each toss was subtracted from the cumulative total of heads. The first three tosses in both series were heads and the fourth toss in the horizontal series was heads, but the fourth toss in the vertical series was tails. Numbers on the dots are the toss numbers. Colored line traces "position" at every fourth toss, just as the track of a Brownian particle recorded by a camera shows only a small fraction of the staggering number of "kicks" such a particle receives from molecules around it.

The second assumption we made was that the arrival times of the two friends are completely independent of each other. This assumption, like the first, is of crucial importance. In mathematical terms it is reflected in the rule of the multiplication of probabilities. This rule states that when individual events are independent of each other, the probability of the complex event that *all* of them will occur is the product of individual probabilities. (Actually from a strictly logical point of view the rule of the multiplication of probabilities constitutes a *definition* of independence.) Independence is assumed in the throw of two dice (which presumably are not linked in any way) as well as in the case of the two suburbanites coming into New York (provided that they have not "coupled" their arrival times by an understanding about the selection of particular trains).

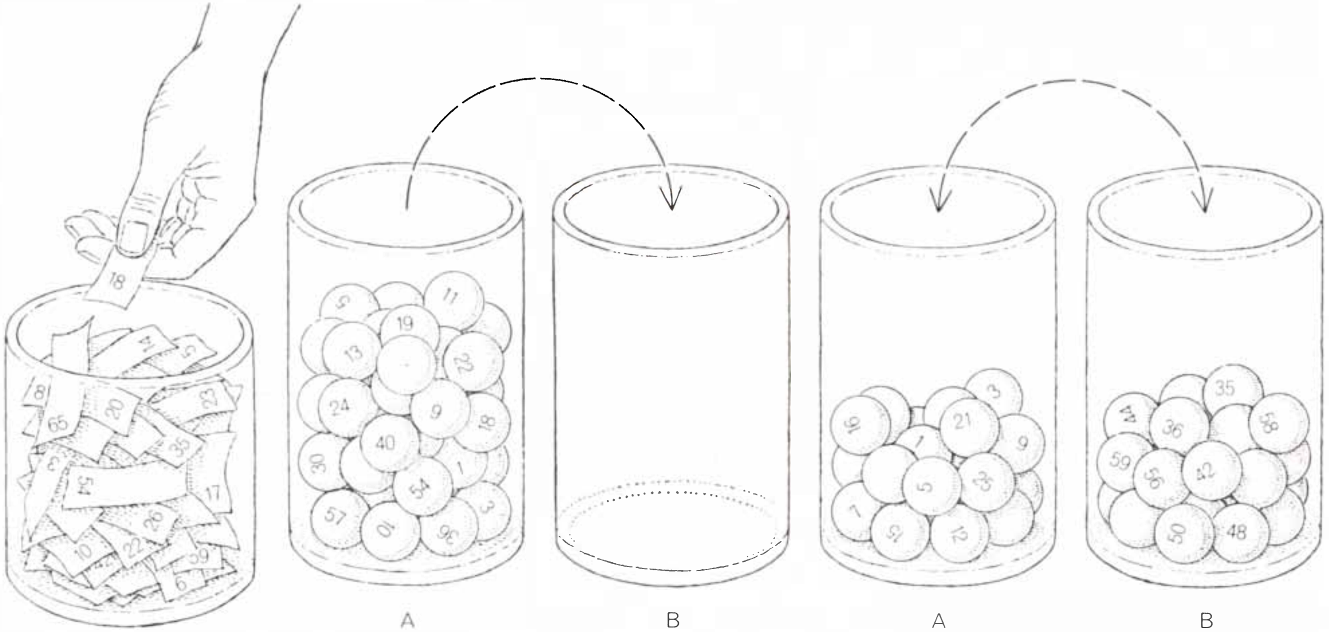
It should be noted that there is an important difference between the dice-throwing and suburbanite-meeting problems. In the first case the number of possible outcomes is finite (just 36), whereas in the second it is infinite, in the sense that the arrival times may occur at any instant within the hour; that is to say, the sample space is a continuum with an infinite number of "points."

To enable one to go on with calculations of probabilities two very general rules, or axioms, are introduced. The first concerns mutually exclusive events: events such that the occurrence of one precludes the occurrence of any other. For such events the probability that at least one will occur is the *sum* of individual probabilities (the axiom of additivity). The second concerns pairs of events such that one *implies* the other. In this case the probability that one will occur but not the other is obtained by subtracting the smaller probability from the larger.

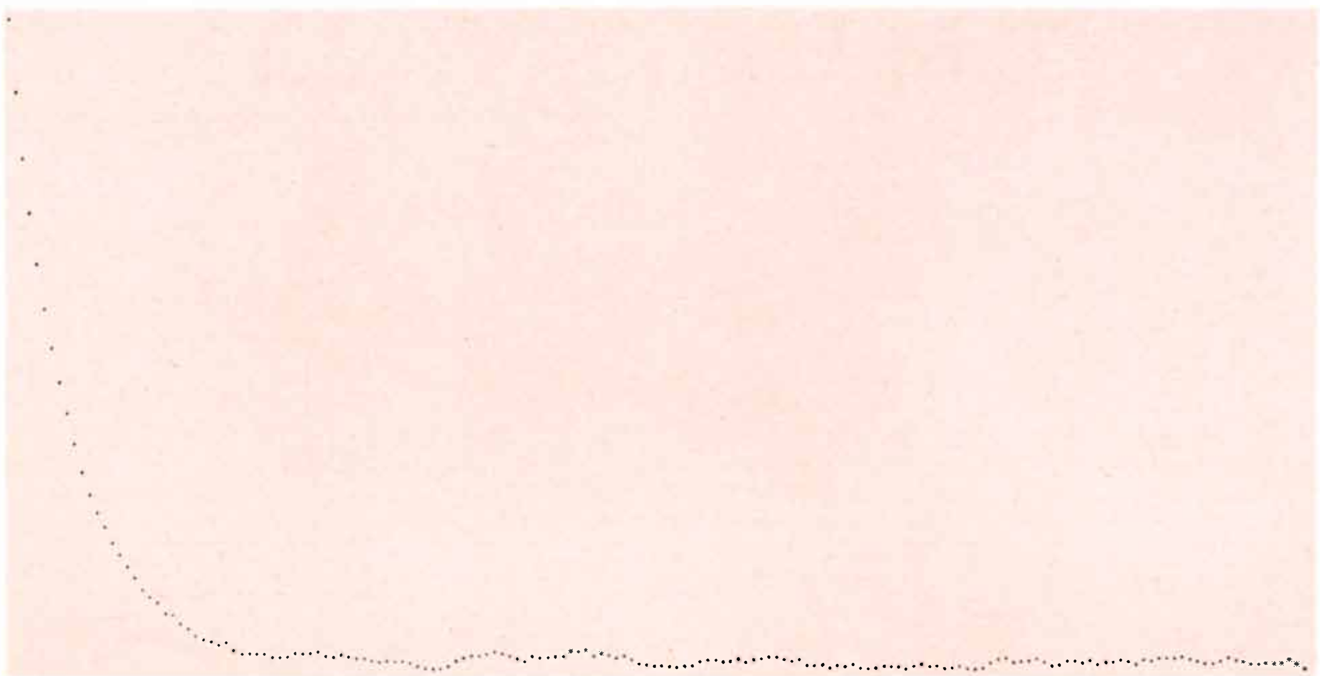
Now, these rules for calculating probabilities of complex events are identical with those used for calculating areas and volumes in geometry. We can substitute the word "set" for "event" and "area" or "volume" for "probability." The problem then is to assign

appropriate areas to sets, and this is the province of measure theory, which has been given that name because the word "measure" is now used to refer to areas of very complex sets.

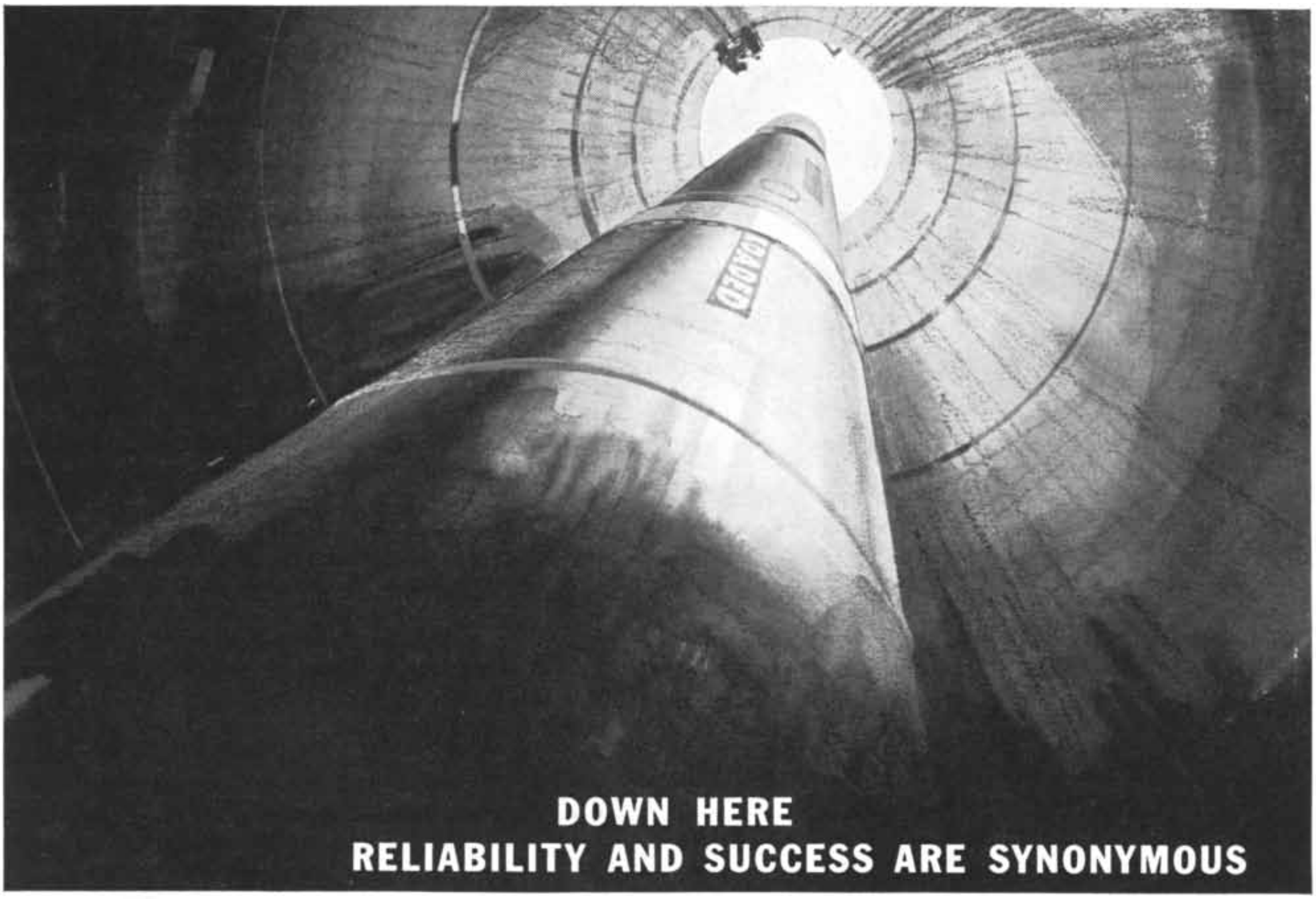
If we go back to the problem of the two suburbanite friends, we note that the set that corresponds to their meeting is quite simple. Its area, or probability, is well within Euclid's framework, and its calculation can be based on the manipulation of only a finite number of nonoverlapping rectangles. In the Buffon needle problem, since the region of interest is bounded by curves, one must allow an infinite number of rectangles, but the calculation of the area is still relatively simple and requires nothing more than elementary calculus. What was surprising and exciting about measure theory as it was de-



EHRENFEST MODEL for illustrating a Markov chain involves a game in which balls are moved from one container to another according to numbers drawn at random from a third container (*left*). As long as there are many more balls in container *A* than in container *B*, the flow of balls will be strongly from *A* to *B*. The probability of finding in *A* the ball with the drawn number changes in a way that depends on past drawings. This form of dependence of probability on past events is called a Markov chain.



PLAYED ON A COMPUTER, an Ehrenfest game with 16,384 hypothetical balls and 200,000 drawings took just two minutes. Starting with all the balls in container *A*, the number of balls in *A* was recorded with a dot every 1,000 drawings. It declined exponentially until equilibrium was reached with 8,192 balls (half of them) in each container. After that fluctuations were not great.



To us this means a requirement for: A diesel electric set for the Environmental Control System of a missile launch control facility, delivering continuous KW for 2500 operational hours—in a three-year period—without service attention of any kind!

This missile assignment meant refining our commercial reliability program, so we could give a specific mathematical probability that a special engine package would perform its mission.

Our research group directed its effort to meet these military requirements by the utilization of data compiled in conjunction with our commercial programs.

Diesel Power and the Bathtub Curve

Prior work included examination of published studies of pre-wearout failures in compound machines. Dr. Morris, of Imperial Chemical Industries, Ltd., had identified two types of failure faults, Stage I and Stage II.

Initially we plotted failure fault curves. These were established by detailed studies of field service reports of engine failures and basic engine test programs charting failure events as they occurred.

Analysis and comparison of these curves with the Bathtub Curve, familiar to reliability people, revealed a mathematical relationship. We then established the bathtub, or failure rate, curve for Caterpillar Engines.

A Predictable Pattern

In our test and study program we found, right at the start, the Stage I faults—the few with high probability of occurrence—did happen as predicted. These were often traced to human error. Some components were predisposed to fail in Stage I. In diesel generator sets, for example, special relays and diodes of 50% more capacity were added to the control circuit. Hose clamps were replaced by sealed flange joints. Gauges were removed completely. Instead, engine inspection kits include a full set of test gauges.

New Designs Required

Ultimately, analysis of potential failure points revealed the need for designing additional capacity oil and air filter systems and an entirely new cooling system. These changes, with attention to detail, were, in terms of the reliability objective, highly rewarding.

With .90 reliability for a 2500-hour mission achieved, Cat Diesel Electric Sets are ready for missile duty.

From this reliability program have come significant benefits to the research for a new VHO engine family for tactical and logistical vehicles. The battlefield reliability standard for these engines was set at .95 for 500 hours—without major breakdown. This Research project for ATAC is already in the building stage.

If you have a vehicle or power problem with a high reliability factor, why not bring it to the people who were successful with the first full-scale reliability effort in the diesel industry? Write or call Defense Products Department, Caterpillar Tractor Co.

CATERPILLAR

Caterpillar and Cat are Registered Trademarks of Caterpillar Tractor Co.

Caterpillar Tractor Co., General Offices, Peoria, Ill. • Caterpillar Americas Co., Peoria, Ill. • Caterpillar Overseas S.A., Geneva • Caterpillar of Australia Pty. Ltd., Melbourne • Caterpillar Brasil S.A., São Paulo • Caterpillar Tractor Co. Ltd., Glasgow • Caterpillar of Canada Ltd., Toronto • Caterpillar France S.A., Grenoble • Caterpillar (Africa) (Pty.) Ltd., Johannesburg veloped by Borel and Lebesgue was that, by merely postulating that the measure of an *infinite* collection of unconnected sets should be the *sum* of the measures of the individual ones (corresponding to requiring that the probability that at least one out of *infinitely many* mutually exclusive events will occur should be the sum of individual probabilities), it was possible to assign measures to extremely complex sets.

Because of this, measure theory opened the way to the posing and solving of problems in probability that would have been unthinkable in Laplace's time. Here, for instance, is one of the problems that received much attention in the 1920's and 1930's and contributed greatly to bringing probability theory into the mainstream of mathematics.

Consider the infinite series

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots$$

This is known as a diverging series; that is, by adding more and more terms one could exceed any given number.

Suppose the signs between the terms, instead of being all pluses, were made plus or minus at random by means of independent tosses of an honest coin. What is the probability that the resulting series would converge? That is to say, what is the probability that by extending the series to more and more terms one would come closer and closer to some terminating number?

To answer the question one must consider all the possible infinite sequences of heads and tails as the sample space. One sequence might begin: *H H T T T H T H*.... If we let *H* represent plus and *T* represent minus, the number series above becomes

$$+\frac{1}{1}+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+\frac{1}{8}\cdots$$

With each such sequence we can associate a real number, t, between 0 and 1, and each t can be represented by a binary number in which the digit 1 denotes *H* and the digit 0 stands for *T*. The sequence cited above is then written as $t = .11000101...$. The binary digits form a model of independent tosses of a coin. Now those t's that will yield convergent series form a set, and the probability that t falls into this set is the "measure" of the set. It turns out that the set of hypothetical t's that do *not* yield convergent series is so sparse that its measure, or probability, is zero (although the set has a very complex structure and is far from being empty). Hence the answer to the problem is that, when the series above is given random signs, the probability that it will converge is 1.

The foregoing is an example of problems in "denumerable probabilities," that is, those involving events described in discrete terms. During the past two decades mathematicians have pursued an even more productive investigation of the theory of "stochastic processes": the probabilistic analysis of phenomena that vary continuously in time. Stochastic processes arise in physics, astronomy, economics, genetics, ecology and many other fields of science. The simplest and most celebrated example of a stochastic process is the Brownian motion of a particle.

The late Norbert Wiener conceived the idea of basing the theory of Brownian motion on a theory of measure in a set of all continuous paths. This idea proved enormously fruitful for probability theory. It breathed new life into old problems such as that of determining the electrostatic potential of a conductor of "arbitrary" shape, a problem that occupied the minds of illustrious mathematicians for more than a century. More than that, it opened up entire new areas of research and led to fascinating connections between probability theory and other branches of mathematics.

A single article can only touch on a few of the main developments and sample problems in probability theory. The subject today embraces vast new fields such as information theory, the theory of queues, diffusion theory and mathematical statistics. One can sum up the position of probability in general by observing that it has become both an indispensable tool of the engineer and a thriving branch of pure mathematics now raised to a high level of formalism and rigor.

I want to close with a brief comment on the philosophical aspect of probability theory (in itself a vast subject on which many volumes have been written). The philosophical implications can be best illustrated by a specific case, and the one I shall discuss has to do with a conflict between the thermodynamic and the mechanical views of the behavior of matter.

Consider two containers, one containing gas, the other a vacuum. If the two containers are connected by a tube and a valve in the tube is suddenly opened, what happens? According to the second law of thermodynamics, gas rushes from container *A* into container *B* at an exponential rate until the pressure in the two containers is the same. This is an expression of the law of increasing entropy, which in its most pessimistic form predicts that ultimately all matter and energy in the universe will even out and settle down to what Rudolf Clausius, one of the fathers of the second law, called *Wärmetod* (heat death).

Now, the mechanical, or kinetic, view of matter pictures the situation in an entirely different way. True, the molecules of gas will tend to move from the region of higher pressure into the one of lower pressure, but the movement is not merely one-way. Bouncing against the walls and against one another, the molecules will take off in random directions, and those that travel into container *B* will be as likely to wander back to container *A* as to remain where they are. As a matter of fact, Poincaré showed in a mathematical theorem that a dynamical system such as this one would eventually return arbitrarily close to its original state, with all or virtually all the gas molecules back in container *A*.

In 1907 Paul and Tatiana Ehrenfest illustrated this idea with a simple and beautiful probabilistic model. Consider two containers, *A* and *B*, with a large number of numbered balls in *A* and none in *B*. From a container filled with numbered slips of paper pick a numeral at random (say 6) and then transfer the ball marked with that number from container *A* to container *B*. Put the slip of paper back and go on playing the game this way, each time drawing at random a number between 1 and *N* (the total number of balls originally in container *A*) and moving the ball of that number from the container where it happens to be to the other container [*see upper illustration on page 102*].

It is intuitively clear that as long as there are many more balls in *A* than there are in *B* the probability of drawing a number that corresponds to a ball in *A* will be considerably higher than vice versa. Thus the flow of balls at first will certainly be strongly from *A* to *B*. As the drawings continue, the probability of finding the drawn number in *A* will change in a way that depends on the past drawings. This form of dependence of probability on past events is called a Markov chain, and in the

game we are considering, all pertinent facts can be explicitly and rigorously deduced. It turns out that, on an averaging basis, the number of balls in container *A* will indeed decrease at an exponential rate, as the thermodynamic theory predicts, until about half of the balls are in container *B*. But the calculation also shows that if the game is played long enough, then, with probability equal to 1, all the balls will eventually wind up back in container *A*, as Poincaré's theorem says!

How long, on the average, would it take to return to the initial state? The answer is 2^N drawings, which is a staggeringly large number even if the total number of balls (N) is as small as 100. This explains why behavior in nature, as we observe it, moves only in one direction instead of oscillating back and forth. The entire history of man is pitifully short compared with the time it would take for nature to reverse itself.

To test the theoretical calculations experimentally, the Ehrenfest game was played on a high-speed computer. It began with 16,384 "balls" in container *A*, and each run consisted of 200,000 drawings (which took less than two minutes on the computer). A curve was drawn showing the number of balls in *A* on the basis of the number recorded after every 1,000 drawings [*see lower illustration on page 102*]. As was to be expected, the curve of decline in the number of balls in *A* was almost perfectly exponential. After the number nearly reached the equilibrium level (that is, 8,192, or half the original number) the curve became wiggly, moving randomly up and down around that number. The wiggles were somewhat exaggerated by the vagaries of the machine itself, but they represented actual fluctuations that were bound to occur in the number of balls in *A*.

Those small, capricious fluctuations are models of the variability in nature and are all that stands between us and the heat death to which we are seemingly condemned by the second law of thermodynamics! Probability theory has reconciled the apparent conflict between the thermodynamic and the kinetic views of nature by showing that there is no real contradiction between them if the second law is interpreted flexibly. In fact, the development of the theory of probability in the 20th century has changed our attitudes to such an extent that we no longer expect the laws of nature to be construed rigidly or dogmatically.

LOOK WHAT

SMOOTHING SPILLWAY of Navajo Dam near Bloomfield, N. M., with bump cutter utilizing 120 natural-diamond blades mounted on a shaft. Spillway must be absolutely smooth, or water vibration will eventually destroy the surface. Bump cutter was held on the 45-degree slope with ⅝-inch cable secured to a winch. Photo courtesy Concut, Inc., El Monte, Calif.

THEY'RE DOING WITH DIAMONDS

INSTALLING RUNWAY LIGHTS

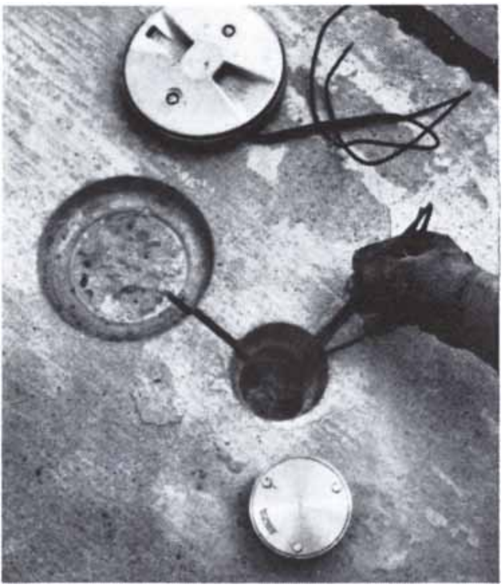
Diamond drills made holes for 700 pancake lights set flush into jet runway at Greater Pittsburgh Airport. Job was performed with core drills made by Wheel Trueing Tool Company, Detroit. Wires for each eight-inch light were attached through saw cut to smaller junction box, then connected to wires coming in from conduit laid before runway was poured.

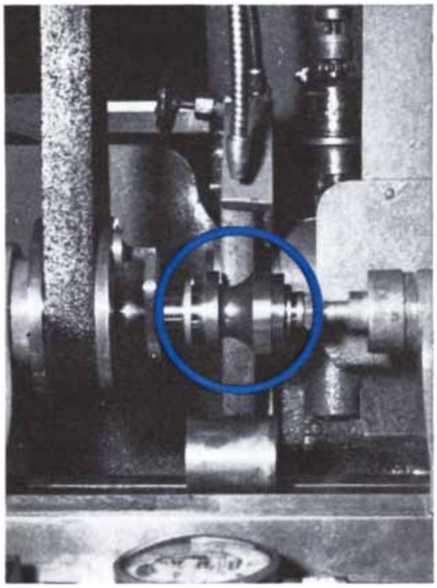
GROOVING NOZZLES

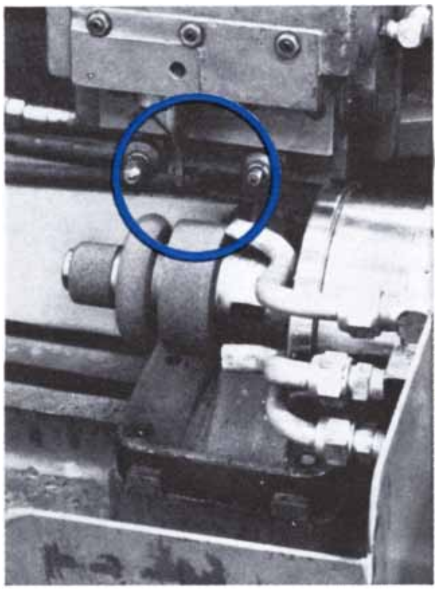
Silicon carbide fuel-injection nozzles for rockets are grooved with diamond-coated wheel made by Dia-Chrome, Inc., Glendale, Calif., at L. A. Gauge, Inc., Sun Valley, Calif. On machine is 1.500-inch-diameter blank. Wheel life in this operation is between 200 and 300 parts, after which wheels are recoated with natural-diamond grit.

DRESSING WHEELS

Three natural diamonds dress three aluminum oxide grinding wheels simultaneously on internal grinder made by the Heald Machine Co., Worcester, Mass. Two fixed diamonds mounted on metal nibs dress straight wheels; a third diamond mounted on radius dresses the radius wheel. The complete dressing operation takes between five and six seconds.







People with imagination are using industrial diamonds for all sorts of intriguing jobs—everything from cutting macaroni to sawing out fossilized dinosaur tracks.

Shown here are four less unusual applications. Although they span several completely different industries, they still share one important detail: each job was done more quickly and more economically than before, with diamonds.

When you use diamonds, you get the unique combination of excellent cutting or grinding ability linked with fantastic endurance. The results: your diamond tools last longer than any other cutting tools you can use. Your people spend more time in actual production . . . less time changing tools.

The Diamond Research Laboratory in Johannesburg, world's leading authority on diamond technology, is constantly engaged in finding new and better uses for industrial diamonds.

If you cut, sharpen or smooth anything in the course of your business, you can probably use diamonds, too. Why not test diamonds against the method you're now using? You'll discover how efficient—and economical—industrial diamonds can be.

INDUSTRIAL DIAMONDS CUT PRACTICALLY EVERYTHING...ESPECIALLY YOUR PRODUCTION COSTS

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The Foundations of Mathematics

When a new mathematical idea is found to be useful, a superstructure of mathematics rises from it. Later the original idea may prove to be shaky, but it must be repaired without destroying the superstructure

by W. V. Quine

Irrefragability, thy name is mathematics. Let natural scientists accept evidence; the mathematician demands proof. Scientific standards have turned austere, it would seem, if anyone is to fuss about foundations for mathematics. Where might he find foundations half so firm as what he wants to found?

Concern for the foundations of mathematics has most often been expressed in emergency situations, when basic ideas begin to seem shaky and mathematicians are forced to examine them. Such an examination was accorded the idea of the infinitesimal long after Isaac Newton and Gottfried Wilhelm von Leibniz developed the differential calculus. This concept of a fractional quantity infinitely close to zero, yet different from zero, provided a foundation for the study of rates, the business of the differential calculus.

Consider a car accelerating from a standstill to a speed of 90 miles an hour. At the instant its speedometer needle points to 60 it is going a mile a minute; at earlier instants its speed is less, at later ones more. The instantaneous speed of a mile a minute does not consist in going a mile in one minute because it is not maintained for a minute. It is not maintained for any time at all by the accelerating car. The distance covered at each instant is zero, and the characterization—no miles an instant—eliminates the distinction between one speed and another. So the founding fathers of the calculus assumed the existence of infinitesimal numbers, just barely distinct from zero and from one another. (We are familiar with smaller and smaller fractions—1/8, 1/16 and so on—but these are not infinitesimals; an infinitesimal is supposed to go into 1 not just 16 times but infinitely many times.)

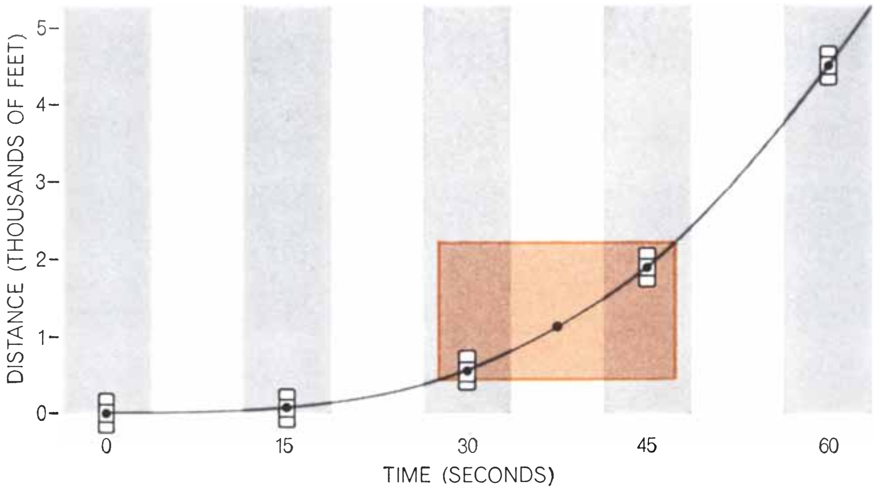
Going a mile a minute, then, means going one of those infinitesimal distances in some infinitesimal time. Going half a mile a minute means going half that infinitesimal distance in that infinitesimal time. The absurdity of this approach was obvious, but the resulting calculus had made it possible to reason mathematically about rates. So a problem arose that is characteristic of problems in the foundations of mathematics: how to get rid of the infinitesimal and make do with clearer ideas while still saving the useful superstructure.

Augustin Cauchy and his followers in the 19th century solved the problem. Consider shorter and shorter intervals of time, each of them straddling our given instant. If over each interval we write the distance the car traveled therein, every distance-to-time ratio will be close to a mile a minute if the time interval is short. Whatever degree of accuracy we care to stipulate, there is a time interval such that for all intervals inside it our distance-to-time ratios will approximate a mile a minute with the stipulated accuracy. A sequence of such distance-to-time ratios, taken over increasingly narrow intervals, approaches a limit (which can be determined by the technique known as differentiation). The notion of limits concerns short but not infinitesimal distances. It can be used to define what it means to be going a mile a minute at a given instant.

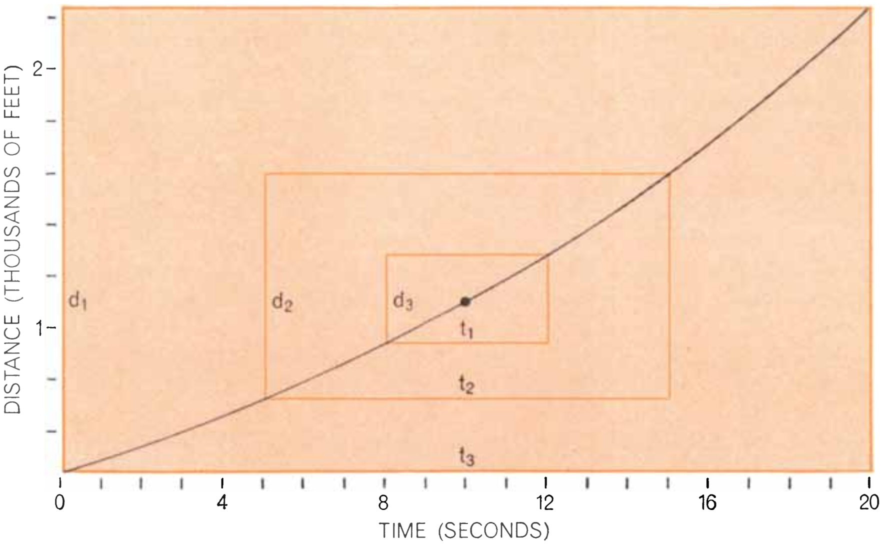
The infinitesimal is not the only mathematical concept that had to be legitimized or eliminated. Take the 16th-century idea, still very much with us, of imaginary numbers: square roots of negative numbers. Square any real number, negative or positive, and the result is positive. What then are the square roots of negative numbers? Whatever they are, they have become so central to applied mathematics that if you so much as divide a time by a distance you end up, according to relativity physics, with an imaginary number. As in the differential calculus, an examination of the foundation must be made with an eye toward preserving the superstructure.

The square root of -1 is the imaginary unit called i. The rest of the imaginary numbers are the multiples of i by real numbers [see "Number," page 50]. Corresponding to the real number 3 there is the imaginary number $3i$; corresponding to the real number 1/2 there is the imaginary $\frac{1}{2}i$; corresponding to the real number π there is the imaginary πi. The imaginary numbers, thus constituted, then combine with the real numbers by addition; we get $3 + i$, $\pi + 2i$ and the like, known as complex numbers, and these impart a utility to the imaginary numbers. Any complex number is a convenient coding or packaging of two real numbers x and y, each of which can be uniquely recovered on demand. This correspondence can be represented on a plane defined by a real x axis and an imaginary y axis [*see*

THE NUMBER 4 is represented on the opposite page as a class containing all 4-member classes. The outer frame is not closed at right because membership in this class is not restricted to the examples shown here. A proper title for the whole design would be "4." To say that a class of tetrahedrons, cones, spheres or cylinders has 4 members implies that it belongs within "4." This view of number is one of several proposed by students of the foundations of mathematics.

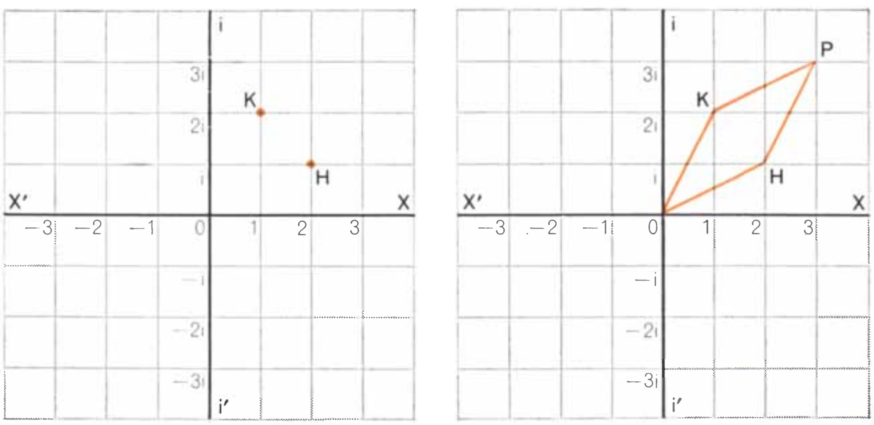


ACCELERATION from a standstill is illustrated. The five vertical strips represent the same one-mile stretch of road viewed at 15-second intervals. In the first 15 seconds the car (*colored object*) advances about 70 feet. In its fourth 15 seconds it moves 32 times farther.



$\frac{d_1}{t_1}=\frac{1{,}799.5 \text{ ft.}}{20 \text{ sec.}}=90 \text{ ft./sec.}$ $\frac{d_2}{t_2}=\frac{884.1 \text{ ft.}}{10 \text{ sec.}}=88.4 \text{ ft./sec.}$ $\frac{d_3}{t_3}=\frac{351.9 \text{ ft.}}{4 \text{ sec.}}=88 \text{ ft./sec.}$

INSTANT at which the car reaches a speed of a mile a minute is a point on the curve of acceleration, a section of which is transposed here. The distance-to-time ratios of ever narrowing intervals will form a sequence that approaches a finite and determinable limit.



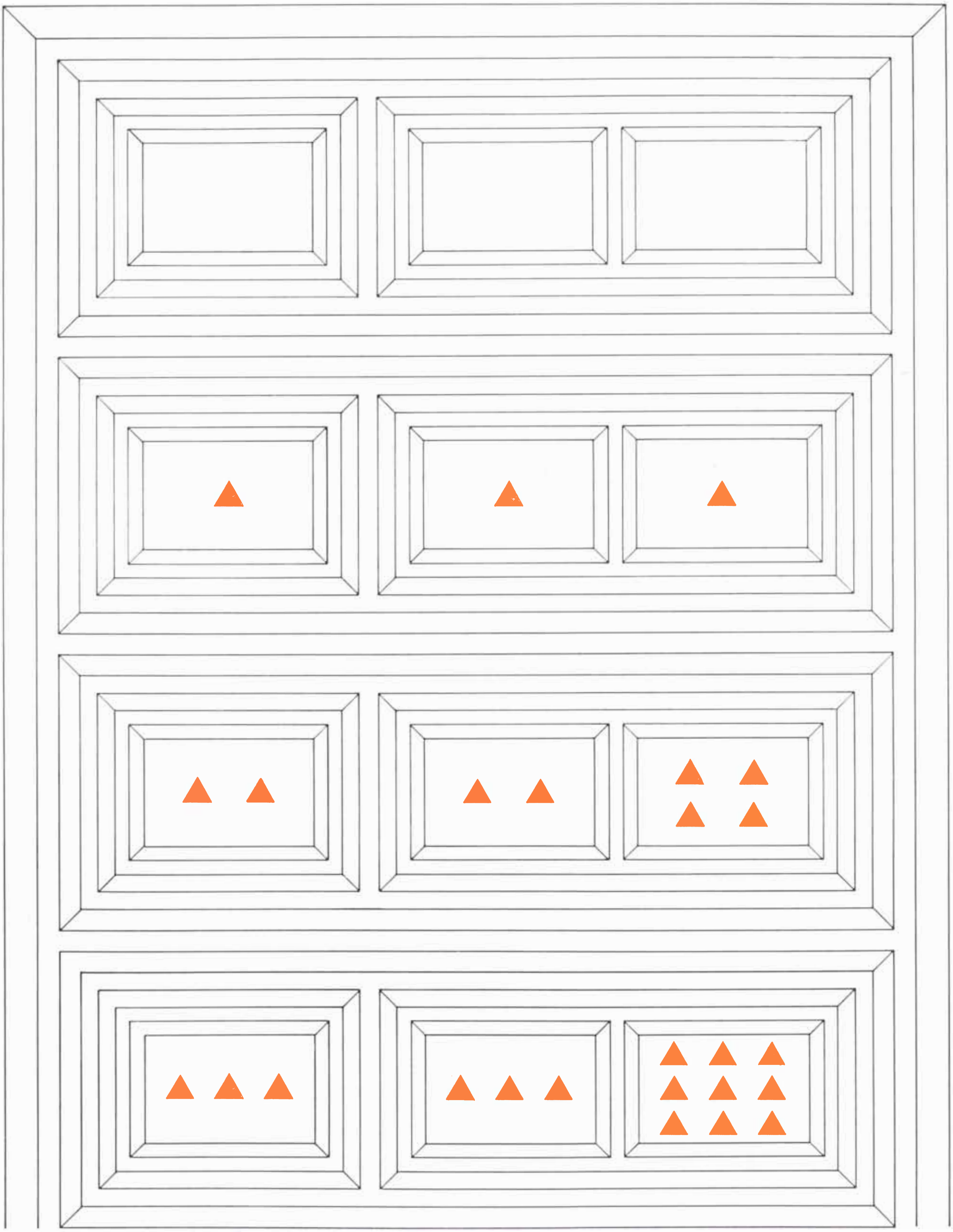
COMPLEX NUMBER $x+yi$ can be plotted on a plane defined by a real horizontal axis and an imaginary vertical axis, as the points $K(1+2i)$ and $H(2+i)$ are plotted at left. If the two points and the origin are vertices of a parallelogram, the fourth vertex (P) is their sum.

bottom illustration on this page]. In retrospect it seems that the mystery of imaginary numbers could have been avoided and the role of complex numbers fulfilled by speaking of "ordered pairs" of real numbers.

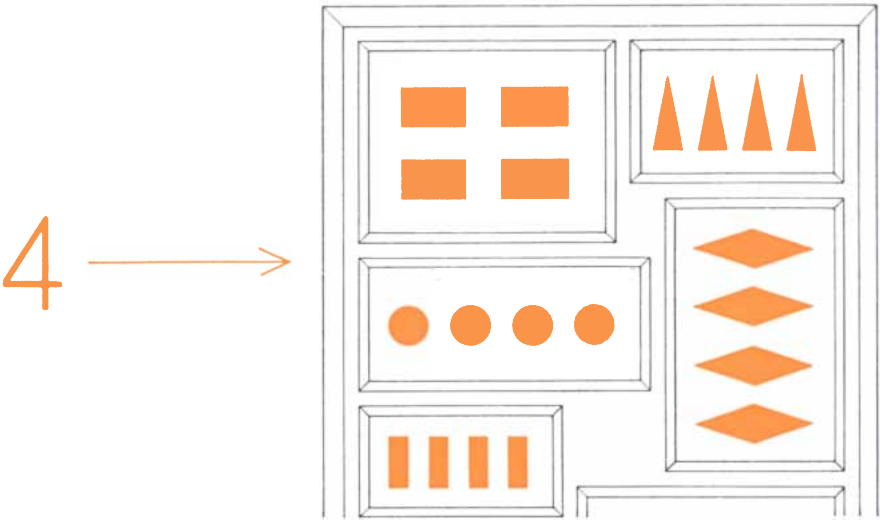
The idea of an ordered pair is helpful at many other junctures in mathematics. Its use is always the same: it is a way of handling two things as one while losing track of neither. Commonly the ordered pair x and y, whether these be numbers or other things, such as fathers and sons, is denoted (x,y). I have not said what things such pairs are, and traditionally the question is skipped; what is important is what they do. Their one property that matters is that if (x,y) is (z,w), then x is z and y is w.

I have said that in principle the myth of imaginary roots could be bypassed. Still, it has value. It greatly simplifies the laws of algebra, an advantage that can be retained while the imaginary and complex numbers are explained away. This is done by a maneuver that is common in foundational studies: *defining* the complex numbers as mere ordered pairs of real numbers and then redefining the usual algebraic operations of plus, times and power so as to make sense of these operations when they are applied to the ordered pairs. The definitions can be so devised that they provide us with an algebra of ordered pairs that is formally indistinguishable from the algebra of complex numbers. One tends to say that the complex numbers have been explained as ordered pairs, but we could just as well say that they have been eliminated in favor of ordered pairs.

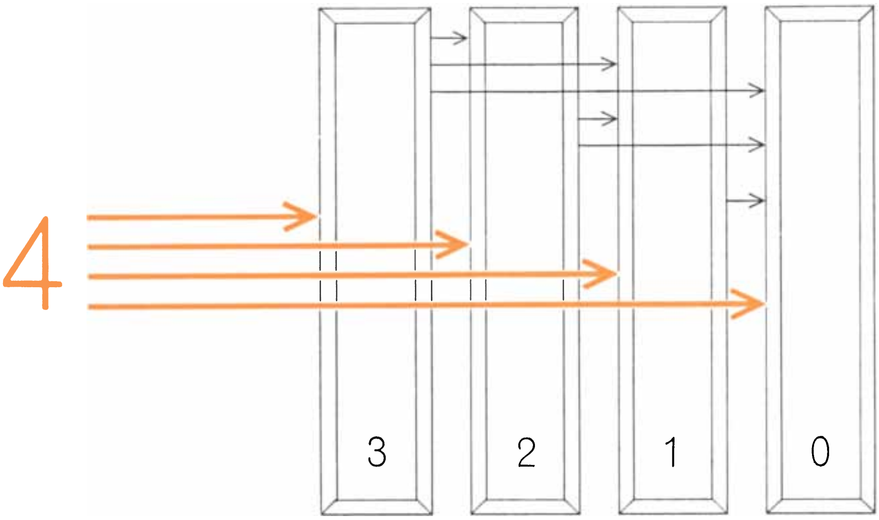
Instead of merely saying what the ordered pairs do, we might go on and try to settle what they are. This question lacks the urgency of the questions about infinitesimals and imaginary numbers, and savors more of casual philosophy. Any answer, however artificial, will serve as long as it upholds the law of pairs: if (x,y) is (z,w), then x is z and y is w. The usual version adopted nowadays comes from Norbert Wiener and Casimir Kuratowski. It does not identify the ordered pair (x,y) simply with the class whose members are x and y; that would confuse (x,y) with (y,x). It identifies (x,y) with a class of two classes. One of the two is the class whose sole member is x; the other is the class whose members are x and y. Here we can say that we have explained ordered pairs as certain two-member classes of classes, or that we have eliminated or-



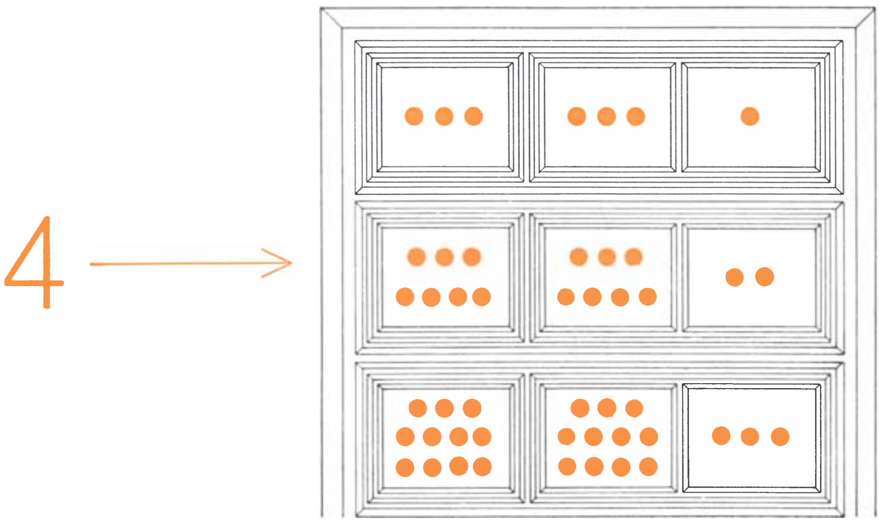
SQUARE-ROOT FUNCTION can be represented by an outer frame unbounded on the bottom, containing ordered pairs (a,b) in which the relation "b is a times as big as a" holds. Each ordered pair is identified as a two-member class. One member class is shown at the left in each frame. *Its* sole member is the first element of the pair. The other member class is shown at the right containing the two elements of the pair. In this convention the bottom pair $(3,9)$ cannot be confused with $(9,3)$, the pair that represents the square function. Credit for this version of pair, which is widely used in modern mathematics, goes to Norbert Wiener and Casimir Kuratowski.



NATURAL NUMBER 4 is shown here as the class of all 4-member classes; thus the outer frame is bottomless. According to this version of number, first suggested by Gottlob Frege, 1 can be explained as the class of those classes that belong to 0 when deprived of a member.



JOHN VON NEUMANN'S VERSION of the natural number 4 emphasizes its ability to pair off in one-to-one correspondence (*indicated by arrows*) with a class of 4 members. Three elements of this class are defined by correspondence. The other is the empty class.



REAL NUMBER 4 is represented as the bottomless class (*outer frame*) of ordered pairs (*a*,*b*) in which *a* is "less than 4 times as big as" *b*. This is the relation that 3 bears to 1, 7 bears to 2 and 11 bears to 3. The convention applies to fractional and negative numbers.

dered pairs in favor of these two-member classes of classes. The difference is verbal, but the first description has the advantage of preserving the notation "(x,y)" and the word "pair."

Philosophical questions seem to lie somewhere between the indignant demands of offended common sense—What is an infinitesimal? What is the square root of a negative number?—and the compulsive questions of a bored child on a rainy Saturday. One philosophical question, deeper than the one about ordered pairs, is: What is number? Let us train this question first on the natural numbers, that is, the positive integers and zero.

Numerals name numbers. The symbol "12" names 12. To rephrase our question: What do the numerals name? What is 12? It is how many Apostles there were, how many months in a year, how many eggs in a carton. But 12 is not merely a property of a dozen eggs, months and Apostles, it is the property common to the class of a dozen eggs, the class of a dozen months and the class of a dozen Apostles.

One of the sources of clarity in mathematics is the tendency to talk of classes rather than properties. Whatever is accomplished by referring to a property can generally be accomplished at least as well by referring to the class of all things that have that property. Clarity is gained because for classes we have a clear idea of sameness and difference; it is a question simply of their having or not having the same members.

In particular, then, we do best to explain 12 not as the property of being a dozen but as the class of all dozens, the class of all 12-member classes. Each natural number n becomes the class of all n-member classes. The circularity of using n thus to define n can be avoided by defining each number in terms of its predecessor. Once we have got to 5, for instance, we can explain 6 as the class of those classes that, when deprived of a member, come to belong to 5. Starting at the beginning, we can explain 0 as the class whose sole member is the empty class; then 1 as the class of those classes that, when deprived of a member, come to belong to 0; then 2 as the class of those classes that, when deprived of a member, come to belong to 1, and so on up.

Just as any version of ordered pairs serves its purpose if it fulfills the law of pairs, so any version of natural number will serve if it fulfills this law: There is a first number and a successor operator that yields something new every

proved improved

This was the proved Type 545A at $1550.

Used by more engineers than any other commercial laboratory oscilloscope, the Type 545A became the standard of the industry.

User suggestions and research innovations helped it grow and develop into the world's best known laboratory oscilloscope—through five years as the Type 545, another five years as the Type 545A.

Over the years, better circuit components and design techniques led to simpler operation and application, greater accuracy and reliability, easier maintenance and calibration.

Seventeen amplifier plug-in units were developed to provide quick adaptability for particular applications. Other features were added or improved to update performance specifications.

With the dual-trace unit, the Type 545A provided 50 mv/cm sensitivity for a wide range of dc-to-24 Mc applications.

Further updating of the "A" Model to implement additional improvements has resulted in a new "B" Model—as the "A" Model was developed from the early Type 545.

So, now, the Type 545A is superseded by the Type 545B. Instrument support will continue to be available for the "A" Model, however, for at least 10 years.

Here is the improved Type 545B at $1550.

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New crt. Internal no-parallax illuminated graticule. Improved resolution, uniform focus over the full 6-cm by 10-cm (50% greater) display area. New hybrid vertical amplifier—greater stability and reliability. Fixed-tuned delay cable, prevents misadjustments. Triggering beyond 30 Mc. Sweep delay, single-sweep, other features and refinements that equal or excel those of the present "A" Model.

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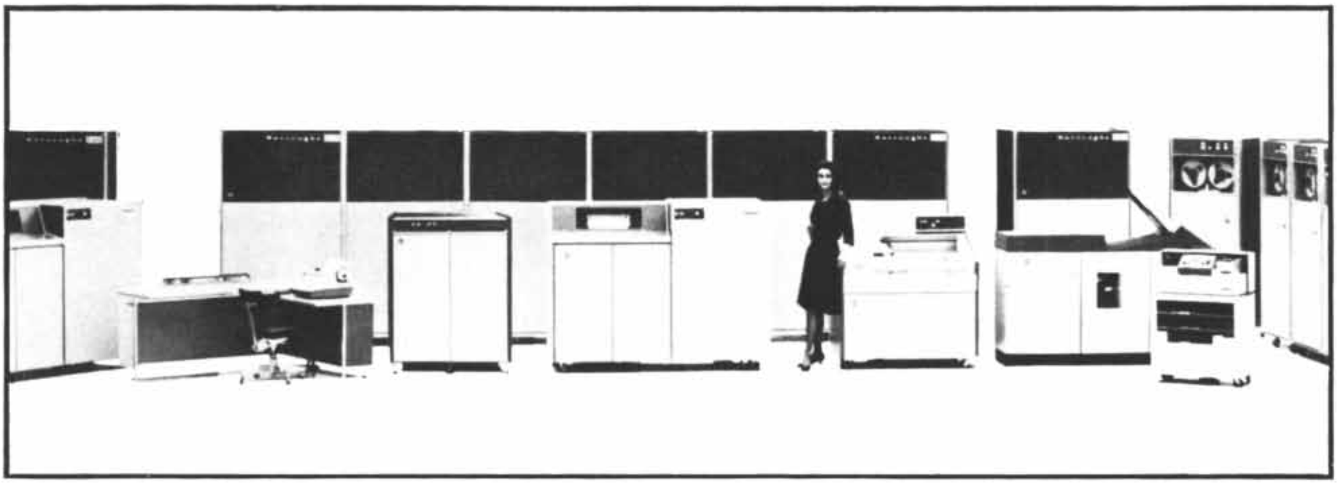
time. The above version of number, presented by Gottlob Frege in 1884, meets the test [*see top illustration on page 116*]. Others do too. A version set forth by John von Neumann identifies each number with the class of all the preceding numbers [*see middle illustration on page 116*]. In this system 0 is itself the empty class; 1 is the class whose sole member is 0; 2 is the class of 0 and 1. Where Frege would say a class of n members belongs to n, von Neumann would say that a class of n members is one whose members can be paired off in one-to-one correspondence with the members of n.

Whether we consider numbers in either of these ways or in some other, a next step is to define the arithmetical operations. The idea behind addition is evident: $m + n$ is how many members a class has if part of it has m members and the rest has n. As for the product $m \times n$, this is how many mem-

STATEMENT ABOUT ***n***	TERMS TO EXPAND	DEFINITION OF TERMS
n is a prime number.	prime number	***n*** is a natural number and, for all natural numbers *h* and *k*, if ***n*** is *h* · *k*, then *h* or *k* is 1.
n is a natural number and, for all natural numbers *h* and *k*, if ***n*** is *h* · *k*, then *h* or *k* is 1.	***n*** is *h* · *k*	A class of ***n*** members falls into *h* parts having *k* members each.
n is a natural number and, for all natural numbers *h* and *k*, if a class of ***n*** members falls into *h* parts having *k* members each, then *h* or *k* is 1.	*x* falls into *h* parts having *h* members each	There is a class *y* of *h* members such that each member of *y* has *k* members and no members of *y* share members and all and only the members of *y* are members of *x*.
n is a natural number and, for all natural numbers *h* and *k*, if for every member *x* of ***n*** there is a member of *y* of *h* such that all members of *y* are members of *k* and no members of *y* share members and all and only the members of the members of *y* are members of *x*, then *h* or *k* is 1.	***n*** is a natural number	***n*** is a member of every class *z* such that 0 is a member of *z* and all successors of members of *z* are members of *z*.
	0	0 is the class whose sole member is the class without members.
	successor	The successor of any *m* is the class of all the classes that, when deprived of a member, come to belong to *m*.
	1	1 is the class of all the classes that, when deprived of a member, come to be the class without members.
n is a member of every class *z* such that the class whose sole member is the class without members is a member of *z* and, for every member *m* of *z* the class of all the classes that, when deprived of a member, come to belong to *m*, is a member of *z* and, for all *h* and *k* that are members of every class *z* such that the class whose sole member is the class without members, is a member of *z* and, for every member *m* of *z* the class of all the classes that, when deprived of a member, come to belong to *m*, is a member of *z*, if for every member *x* of ***n*** there is a member *y* of *h* such that all members of *y* are members of *k* and no members of *y* share members and all and only the members of the members of *y* are members of *x*, then *h* or *k* is the class of all the classes that, when deprived of a member, come to be the class without members.		

SIMPLIFIED SENTENCE appears at bottom left in this chart. Defining the statement "*n* is a prime number" creates a vocabulary of terms (*middle column*) that must be expanded (*as shown in column at right*). The final sentence deals only with class membership. It could be rewritten using only the locutions "and," "not," "is a member of," and the idiom of universal quantification, "everything *x* is such that . . . *x* . . . ," if brevity were no object. The version of number used in the expansion of terms was developed by Frege.

BREAKTHROUGHPUT



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bers a class has if it falls into m parts having n members each.

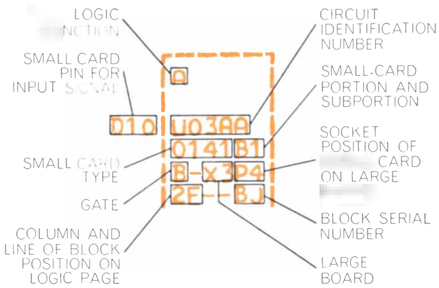
There are the negative numbers still to account for, and the fractional ones, and all the irrational numbers such as $\sqrt{2}$ and π; in short, all the real numbers except the natural numbers. Here again any version will serve that meets certain requirements. One overall version, with more unity than most, construes each real number as being a certain relation between natural numbers; in fact, a certain relation of comparative size. Take in particular the real number 1/2. It is identified as the relation that 1 bears to each integer from 3 on, and that 2 bears to each from 5 on, and so forth. Similarly, each positive real number x is identified with the relation "less than x times as big as." The real number $1/\pi$, for instance, comes out as a relation that 1 bears to each integer from 4 on, and 2 bears to each from 7 on, and 3 to each from 10 on, and so forth. As for the negative real numbers, these are taken as the converse relations; since 1/2 is the relation "less than half as big as," −1/2 comes out as the relation "more than twice as big as."

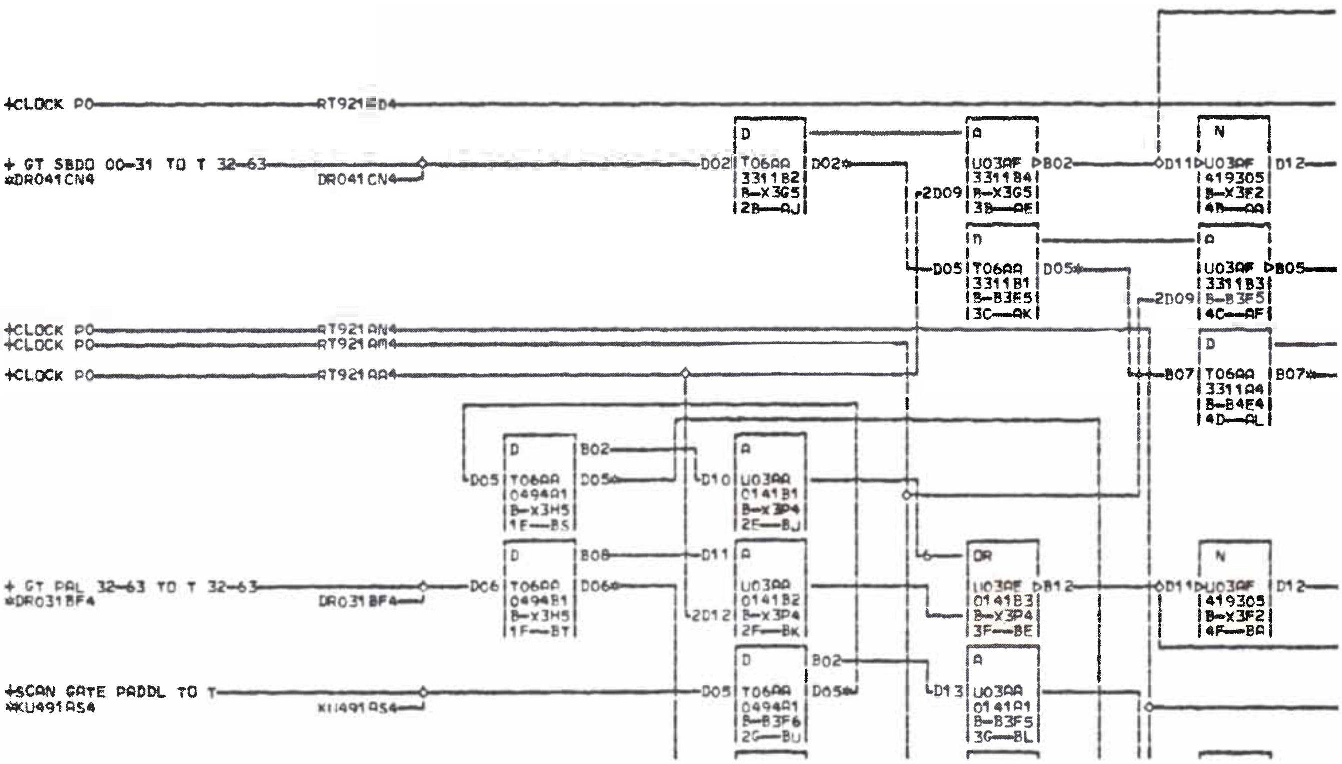
At the risk of sounding increasingly like the child on a rainy Saturday we may interject: But what is a relation? As suggested in our discussion of ordered pairs, a relation can be identified with the class of all the ordered pairs (a,b) such that a bears the relation to b. In this way the real numbers are ultimately identified with classes, as were the natural numbers. The number 1/2 becomes the class of all ordered pairs (a,b) such that a is less than half as big as b. Whereas the class of ordered pairs identified with 1/2 cannot contain a pair (2,4) without violating the "less than" relation—a would be exactly x times as big as b—it could contain the ordered pair (20,41) or any ordered pair where a is within any desired proximity of x times as big as b.

Each real number then corresponds to a distinct class of ordered pairs. This distinctness can be proved by the existence of a rational number between any two points on a line of real numbers. That is, if x and y are different real numbers (say x is less than y), then there is a rational number a/b (a and b are integers) such that a/b is less than y but greater than x. Then the ordered pair (a,b) will fall into the class corresponding to y but not into the class corresponding to x, thus distinguishing the two.

The description of the positive real number x as the relation "less than x times as big as" falls into the same circularity we noticed in describing n as the class of all n-member classes. But now as then the description helps us to see what objects the numbers are to be. The reason it serves is that the circular reuse of "n" or "x" inside the description has a commonsense context. Actually the circularity can be eliminated in both cases by means of careful and complex definition.

We had to adopt a version of the natural numbers before construing the real numbers in general because we took these as being relations of natural numbers. The natural numbers must therefore be seen as distinct, however uselessly, from the corresponding whole real numbers. The real number 5, for example, comes out as the class of ordered pairs (a,b) of natural numbers





LOGIC DIAGRAM shows the design of a computer system in terms of connected elements known as "and/or blocks." The information associated with a block is given in the detail of an "and" block at top left. The logic functions *A*, *OR* and *N* in the upper left-hand corners of the colored blocks denote the logical locutions "and," "or" and "not." The diagram is based on the IBM 360 system.

such that a is less than 5 times as big as b.

In mathematics, talk of functions is no less plentiful than talk of number. But the philosophical question—What are functions?—is more quickly settled than that of number. A function can be identified with the relation of its values to its arguments. The function "square root of" can be explained as the relation of root to square: the relation that 0 bears to 0, that 1 bears to 1, that 2 bears to 4, that 2/3 bears to 4/9 and so on. Thus the square-root function becomes the class of all the pairs (0,0), (1,1), (2,4), (2/3,4/9), in general (x,x^2). The illustration on page 115 represents this function as a class.

Our sample studies in the foundations of mathematics started with trouble-shooting and have leveled out into a general tidying up. It is a process, we see, of reducing some notions to others and thereby diminishing the inventory of basic mathematical concepts. Let us apply this technique in reducing a familiar notion, the definition of a prime number, into elemental class terms. To keep track of our presuppositions we must render our successive definitions in explicit detail and take note of all the logical and mathematical devices that are used in them. Each definition must explain how to eliminate some locution by paraphrasing it, or the sentences in which it appears, into a residual vocabulary that eventually narrows to rudimentary terms. We begin by changing

> n *is a prime number*

to

> n *is a natural number and for all natural numbers* h *and* k, *if* n *is* h × k, *then* h *or* k *is 1.*

Our first step eliminated the troublesome "prime number" from the residual vocabulary but left in its place "n is a natural number," the notation for multiplying h by k and the notation for 1. We know how to eliminate the multiplicative notation by expanding the clause $n = h \times k$ to read

> *a class of* n *members falls into* h *parts having* k *members each.*

To replace the notion of "parts" of a class with the simpler concept of membership, this clause can be changed to

> *for every class* x *with* n *members there is a class* y *of* h *members such that each member of* y *has* k *members and no members of* y *share members and all and only the members of the members of* y *are members of* x.

Cumbersomeness is increasing apace, but the vocabulary is being reduced to terms of class membership. We can now eliminate "x has n members" and kindred clauses. If we use Frege's version of the natural numbers, this becomes "x is a member of n." Our original phrase has now been analyzed to mean

> n *is a natural number and, for all natural numbers* h *and* k, *if for every member* x *of* n *there is a member* y *of* h *such that all members of* y *are members of* k *and no members of* y *share members and all and only the members of the members of* y *are members of* x, *then* h *or* k *is 1.*

The dwindling of perspicuity is less noteworthy than the reduction of vocabulary. Where perspicuity is to our purposes the eliminated locutions can be restored, after all, as defined abbreviations.

The term that next calls for elimina-

NUMBER	STEP	SOURCE
1	$x = x - (y - y).$	AXIOM
2	$x - (y - z) = z - (y - x)$	AXIOM

(From these we derive theorems by substituting any one term for all occurrences of any variable or, given an equation, substituting its one side for its other side anywhere.)

3	$z = z - (y - y)$	Step 1
4	$z = z - (x - x)$	Step 3
5	$y - y = (y - y) - (x - x)$	Step 4
6	$x - (x - z) = z - (x - x)$	Step 2
7	$x - (x - (y - y)) = (y - y) - (x - x)$	Step 6
8	$x - x = (y - y) - (x - x)$	Steps 1, 7
9	$x - x = y - y$	Steps 5, 8

(The expression "$x + y$" can be defined as short for "$x - ((y - y) - y)$." The laws of addition then come through as mere abbreviations of laws of subtraction. Thus the law "$x + y = y + x$" is proved as follows.)

10	$x - ((x - x) - y) = y - ((x - x) - x)$	Step 2
11	$x - ((y - y) - y) = y - ((x - x) - x)$	Steps 9, 10
12	$x + y = y + x$	Step 11, definition

PROOF PROCEDURE for arithmetical addition and subtraction begins with equations (*Steps 1 and 2*) taken as axioms. Then the steps for deriving theorems by substitution are defined. Steps 3 through 9 follow from previous steps listed in the column at right. Addition is defined in terms of subtraction before its elementary law is proved (*Steps 10–12*).

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tion is "is a natural number" (said of *n, h* and *k*). To say that *n* is a natural number is to say that *n* is 0 or successor of 0 or successor of that or so on. Frege showed how to dodge the "so on" idea by defining "a natural number" as

a member of every class z *such that 0 is a member of* z *and all successors of members of* z *are members of* z.

Frege explained "0" as the class whose sole member is the class without members, and the "successor" of any *m* as the class of all the classes that, when deprived of a member, come to belong to *m*. If we eliminate "0" and "successor" in rewriting the above version of "is a natural number" and then use the result as a clause in rewriting our original phrase, we end up with a long story in a short vocabulary [*see illustration on page 118*]. The number "1" at the end of our intermediate definition is resolved too, because 1 is successor of 0. The vocabulary that remains refers to class membership and little else. There is an assortment of elementary logical particles: "is," "and," "or," "if-then," "every," "all" and the like.

By further steps, all of these can be reduced to several basic locutions. One is "and" as a connective of sentences. Another is "not." A third is the idiom of universal quantification, "everything is such that ... it ..." or, more flexibly, "everything *x* is such that ... *x* ...," with variable letters. The prefix "everything *x* is such that" is compactly symbolized as "(*x*)." Fourth, there is the transitive verb "ϵ" meaning "is a member of." Perhaps the list should also include, fifth, the parentheses used to group clauses. The following, then, is an illustrative, brief sentence in our frugal notation:

(x) *not* (y) *not* (x ϵ y *and not* y ϵ x).

It amounts in effect to the words "Everything is a member of something not a member of it."

Every sentence expressible in the notation of pure classical mathematics, whether in arithmetic or the calculus or elsewhere, can be paraphrased into this thumbnail vocabulary, if not with comparable brevity. What "*n* is a prime number" was seen to stretch to is terse compared with what it would be if fully paraphrased in our five basic idioms. The five are not to be recommended as a *lingua franca* of mathematics, nor as a practical medium of computation. But it is of theoretical interest that so much in the way of mathematical ideas can be generated from so meager a basis, and from this basis in particular.

Four of the five basic locutions belong to logic. One, "ϵ," is peculiar to set theory, or the mathematics of classes. Or we might say that all five are in set theory; the logical locutions are a proper adjunct of every science, after all, and so of set theory.

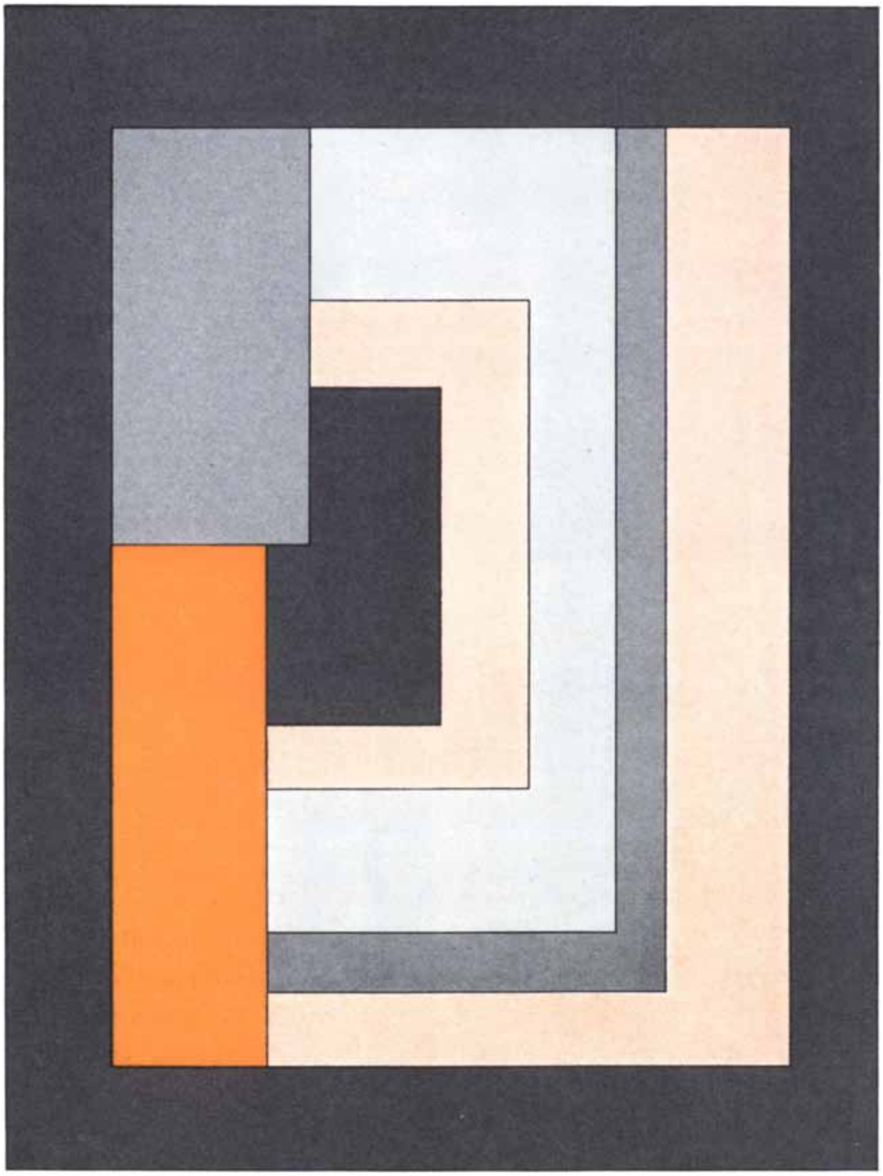
It would appear, then, that all mathematics can be paraphrased in a set-theory vocabulary. Therefore all mathematical truth can be seen as truth of set theory. Every mathematical problem can be transformed into a problem of set theory. Either this augurs well for the outstanding problems of mathematics or else set theory has problems as deep as those of classical mathematics.

The latter is the case. And the worst aspect of set theory is not just that sentences can be written whose truth or falsity is hard to prove, but that sentences can be written whose simultaneous truth and falsity seem all too easy to prove. One such is the sentence

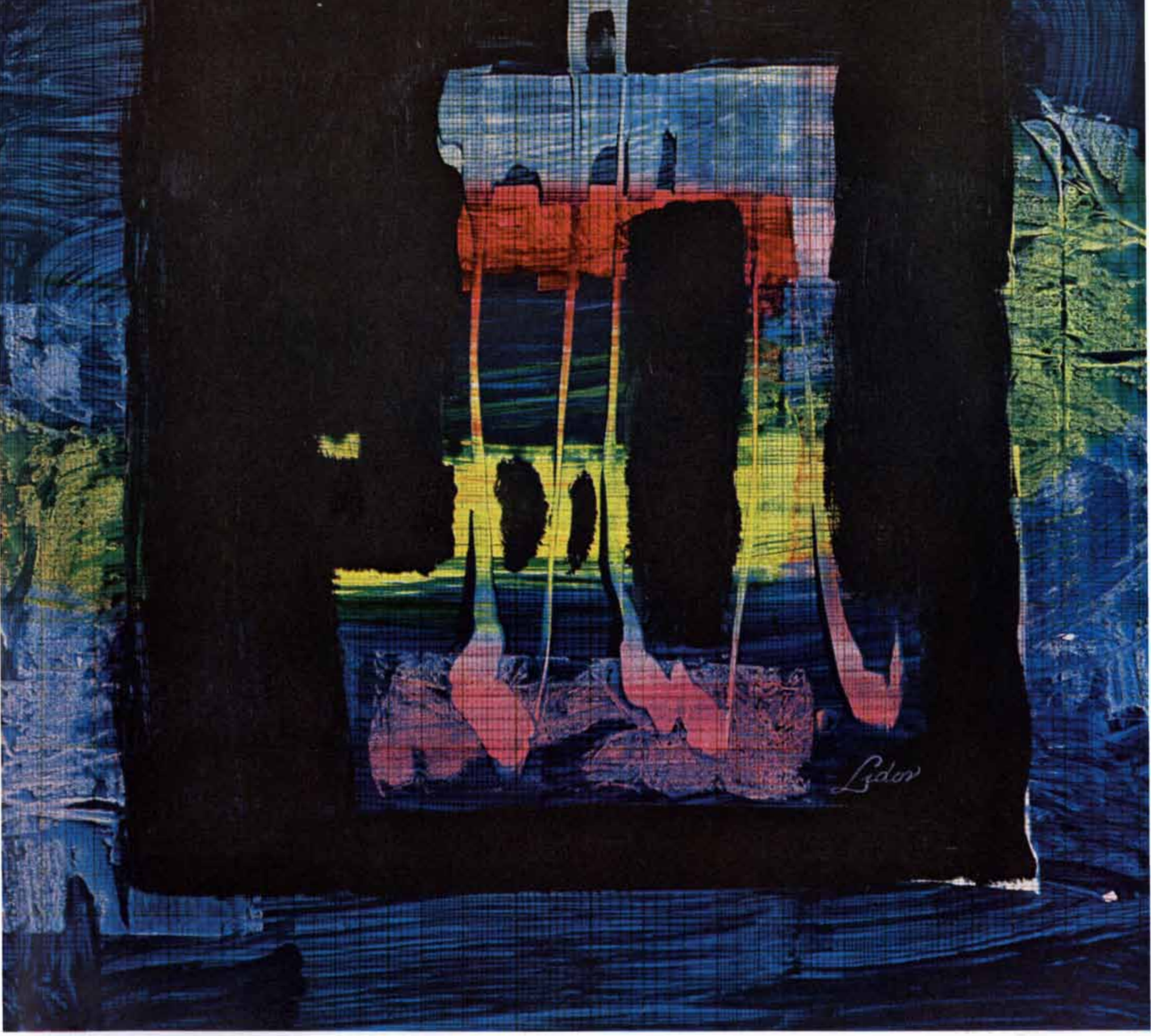
not (y) *not* (x) [*not* (x ϵ y *and* x ϵ x) *and not* (*not* x ϵ y *and not* x ϵ x)].

Partially transcribed with an eye to mortal communication, it says:

There is something y *such that* (x)



FIVE-COLOR MAP is uneconomical and can be redrawn using only four colors so that no two countries of like color share a border. No mathematical proof of this statement has been found for the general case, yet no one has drawn a map that will require five colors.



Helping people to help themselves

In the past ten years, the population of the underdeveloped countries has increased by 250,000,000. In many cases, despite heroic efforts to improve both agricultural and industrial technology, these countries are actually losing ground in terms of per capita production—including food. Foreign aid helps, of course. But it is not a final answer.

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(x ϵ y *if and only if not* x ϵ x).

It seems to be true; just take y as the class of all the things x such that x is not a member of itself. Yet it must be false; if y were as averred, we could take x in particular as y and conclude, in self-contradiction, that $y \in y$ if and only if not $y \in y$.

This paradox, turned up by Bertrand Russell in 1901, is the simplest of many in set theory. The moral of them all is that giving a necessary and sufficient condition for membership in a class does not guarantee that there is such a class. Russell's paradox shows in particular that there is no class of exactly the things that are not members of themselves. Consequently the great task of set theory comes to be that of deciding what classes there are. No natural and tenable answer is known; what seemed the natural answer, that there is a class for every membership condition, is untenable.

Since 1901 there has been a proliferation of set theories, no one of them clearly best. Even the question of freedom from self-contradiction is moot in such a framework, since we no longer can trust common sense for the plausibility of the propositions. Common sense in set theory is discredited by the paradoxes. As a foundation for mathematics, set theory is far less firm than what we have founded on it.

Clearly we must not look to the set-theoretical foundation of mathematics as a way of allaying misgivings regarding the soundness of classical mathematics. What we are looking for, as we evaluate various plans for a workable set theory, is a scheme that will reproduce in the eventual superstructure the accepted laws of classical mathematics. We find ourselves regarding set theory as a conveniently restricted vocabulary in which to formulate a general axiom system for classical mathematics—let the sets fall where they may.

Such a program of axiomatization can never be completed. There is no hope of a proof procedure strong enough to cover all the truths of classical mathematics, or even of arithmetic, while excluding all the falsehoods. This remarkable fact was proved by Kurt Gödel in 1931.

The proof procedure for addition and subtraction shown in the illustration on page 122 is *complete;* every truth that can be expressed in the notation can be proved by the procedure. This notation, however, covers only a few aspects of arithmetic, neglecting multiplication as well as the logical operators. If notations were added for these further purposes, then no proof procedure could cover all the expressible truths and avoid the falsehoods. This is the case even if we limit the values of the variables to natural numbers.

Such is the notation of what is called elementary number theory. A typical truth in this notation is

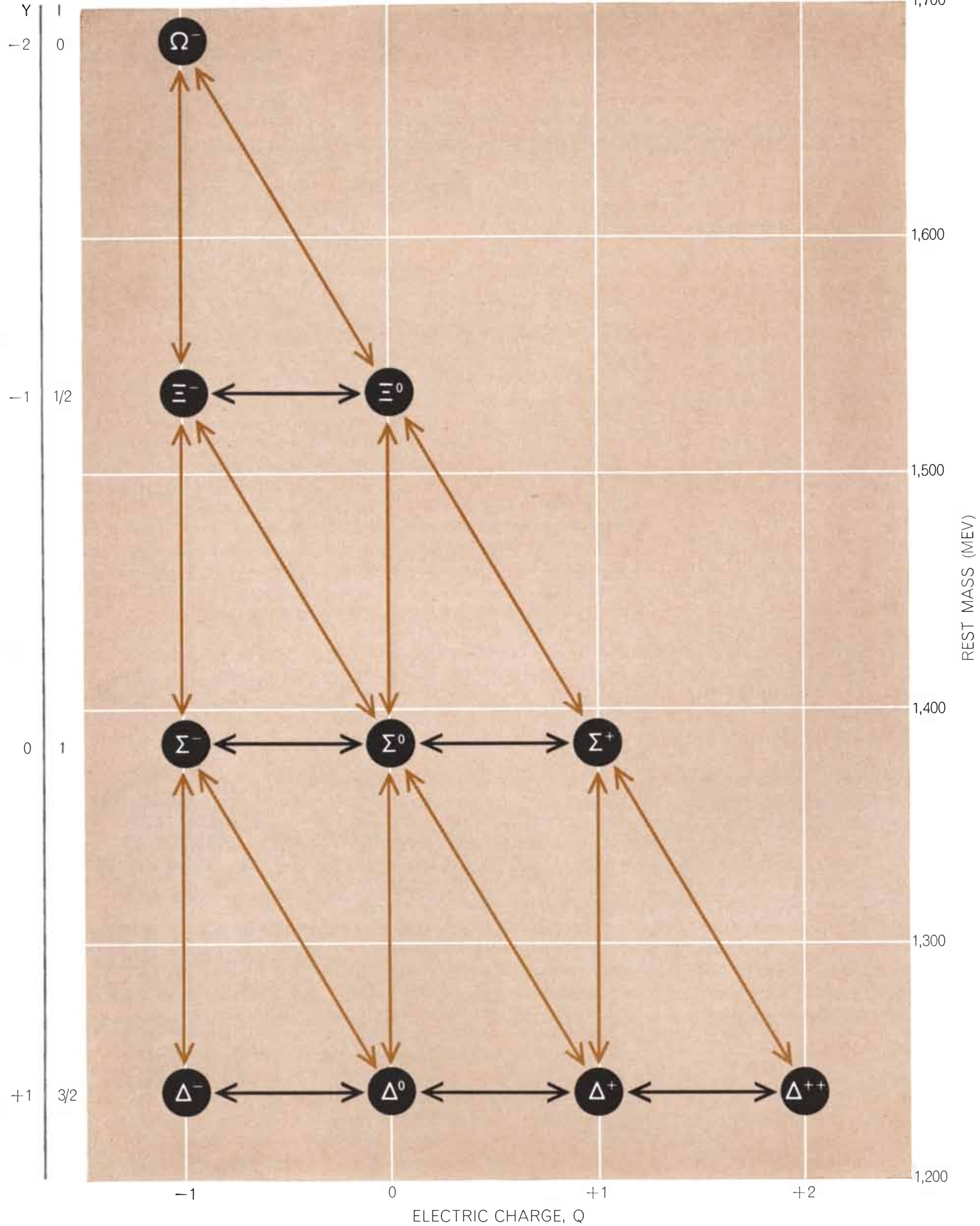
(x) (y) *not* (z) [*not* (x = y + z) *and not* (y = x + z)].

This amounts to saying of all natural numbers x and y that either $x = y + z$ or $y = x + z$ for some natural number z. Gödel showed that given a proof procedure we can construct a sentence in this meager notation that is false if it admits of proof under that procedure, and true if it does not. Therefore, Gödel concluded, our given procedure is either unsound, since it affords proof of a falsehood, or else incomplete, since it fails to afford proof of a truth of elementary number theory.

Gödel's discovery was a shock to preconceptions. The very nature of mathematical truth, one supposed, was its demonstrability. But not so. Surely each sentence constructible in this limited and lucid notation of elementary number theory is significant, each is true or false, each or its own negation is true; yet its truth does not assure demonstrability. The difference between truth in mathematics and truth in natural science is perhaps less abrupt than we thought.

Work in the foundations of mathematics can be concerned with concepts and it can be concerned with laws. Work with concepts—the reduction of concepts by defining some in terms of others—occupied us throughout much of this article. But it is to the study of laws and their encapsulation in axioms or proof procedures that Gödel's discovery relates. This type of work was not curtailed by the realization that we cannot get complete systems for substantial branches of mathematics; we *can* get incomplete ones that are illuminating in various ways.

The fact is that Gödel's result has greatly stimulated work in the branch of foundation studies devoted to laws. The remarkable techniques that went into Gödel's proof have brought about an imposing and thriving branch of mathematics: proof theory. Here is an instance where the foundation did give rise to the superstructure.



SUCCESS OF GROUP THEORY in the physics of fundamental particles was dramatized early this year with the discovery of the omega-minus (Ω^-) baryon at the Brookhaven National Laboratory. The existence of the omega minus had been predicted by the "eightfold way," a theory devised independently by Murray Gell-Mann and Yuval Ne'eman. The term "eightfold" refers to a classification scheme based on the mathematical theory of abstract groups. Previous theory had shown that "isotopic spin" symmetry (*black arrows*) connects families of particles with different values of electric charge (Q). The eightfold way invokes a new system of symmetries (*colored arrows*) to group together superfamilies of particles with different values of hypercharge (Y) and isotopic spin (I). The omega-minus baryon was needed to complete a superfamily of 10 members, of which nine members were previously known: a delta (Δ) quartet, a sigma (Σ) triplet and a xi (Ξ) doublet. The omega minus is the only baryon singlet with a negative electric charge, and its observed mass is within a few million electron volts (mev) of the mass predicted by theory.

Mathematics in the Physical Sciences

The outstanding examples of the power of mathematics to relate the facts of nature can be found in physics. The latest example is the use of group theory to relate the fundamental particles

by Freeman J. Dyson

In 1910 the mathematician Oswald Veblen and the physicist James Jeans were discussing the reform of the mathematical curriculum at Princeton University. "We may as well cut out group theory," said Jeans. "That is a subject which will never be of any use in physics." It is not recorded whether Veblen disputed Jeans's point, or whether he argued for the retention of group theory on purely mathematical grounds. All we know is that group theory continued to be taught. And Veblen's disregard for Jeans's advice turned out to be of some importance to the history of science at Princeton. By an irony of fate group theory later grew into one of the central themes of physics, and it now dominates the thinking of all of us who are struggling to understand the fundamental particles of nature. It also happened by chance that Hermann Weyl and Eugene P. Wigner, who pioneered the group-theoretical point of view in physics from the 1920's to the present, were both Princeton professors.

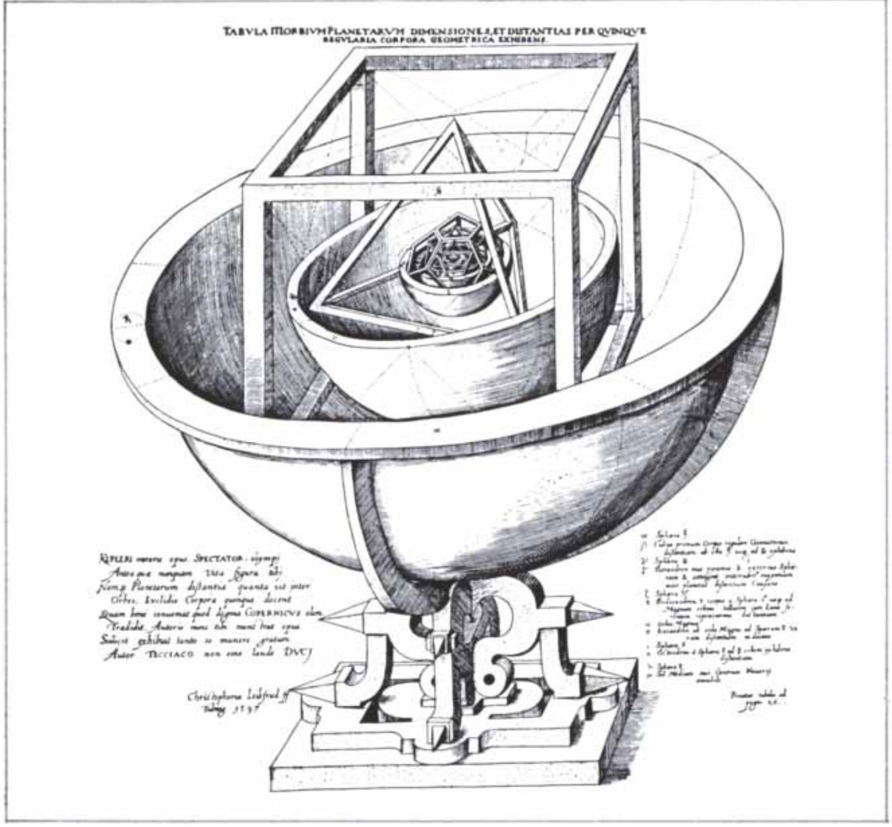
This little story has several morals. The first moral is that scientists ought not to make off-the-cuff pronouncements concerning matters outside their special field of competence. Jeans provides us with a clear lesson on the evil effects of the habit of pontification. Starting from this unfortunate beginning with Veblen, he later went from bad to worse, becoming a successful popular writer and radio broadcaster, accepting a knighthood and ruining his professional reputation with suave and shallow speculations on religion and philosophy.

We ought not, however, to look so complacently on the decline and fall of Jeans. There, but for the grace of God, go we. After all, Jeans in 1910 was a respected physicist (although Princeton, aping the English custom in titles as in pseudo-Gothic architecture, called him professor of applied mathematics). He was neither more incompetent nor more ignorant than most of his colleagues. Very few men at that time had the slightest inkling of the fruitfulness that would result from the marriage of physics and group theory. So the second and more serious moral of our story is that the future of science is unpredictable. The place of mathematics in the physical sciences is not something that can be defined once and for all. The interrelations of mathematics with science are as rich and various as the texture of science itself.

One factor that has remained constant through all the twists and turns of the history of physical science is the decisive importance of mathematical imagination. Each century had its own particular preoccupations in science and its own particular style in mathematics. But in every century in which major advances were achieved the growth in physical understanding was guided by a combination of empirical observation with purely mathematical intuition. For a physicist mathematics is not just a tool by means of which phenomena can be calculated; it is the main source of concepts and principles by means of which new theories can be created.

All through the centuries the power of mathematics to mirror the behavior of the physical universe has been a source of wonder to physicists. The great 17th-century astronomer Johannes Kepler, discoverer of the laws of motion of the planets, expressed his wonder in theological terms: "Thus God himself was too kind to remain idle, and began to play the game of signatures, signing his likeness into the world; therefore I chance to think that all nature and the graceful sky are symbolized in the art of geometry." In the more idealistic 19th century the German physicist Heinrich Hertz, who first verified James Clerk Maxwell's electromagnetic equations by demonstrating the existence of radio waves, wrote: "One cannot escape the feeling that these mathematical formulae have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than was originally put into them." Lastly, in our rationalistic 20th century Eugene Wigner has expressed his puzzlement at the success of more modern mathematical ideas in his characteristically dry and modest manner: "We are in a position similar to that of a man who was provided with a bunch of keys and who, having to open several doors in succession, always hit on the right key on the first or second trial. He became skeptical concerning the uniqueness of the coordination between keys and doors."

The mathematics of Kepler, the mathematics of Hertz and of Wigner have almost nothing in common. Kepler was concerned with Euclidean geometry, circles and spheres and regular polyhedra. Hertz was thinking of partial differential equations. Wigner was writing about the use of complex numbers in quantum mechanics, and no doubt he was also thinking about (but not mentioning) his own triumphant introduction of group theory into many diverse areas of physics. Euclid, partial differential equations and group theory are three branches of mathematics so remote from each other that they seem to belong to different mathematical universes. And yet all three of them turn out to be intimately involved in our one physical universe. These are astonishing facts, understood fully by nobody. Only one conclusion seems to follow



KEPLER'S MODEL OF THE SOLAR SYSTEM, published in 1596, was based on the five "perfect" solids of Euclidean geometry. The planetary orbits were successively inscribed in and circumscribed about an octahedron, an icosahedron, a dodecahedron, a tetrahedron and a cube. The model is a supreme example of misguided mathematical intuition. Although Kepler was aware of the discrepancies between his theory and the best observations of his time, he always regarded this model as one of his greatest achievements.

with assurance from such facts. The human mind is not yet close to any complete understanding of the physical world, or of the mathematical world, or of the relations between them.

In this article I shall not attempt any deep philosophical discussion of the reasons why mathematics supplies so much power to physics. In each century it is only a few physicists—in our century perhaps only Albert Einstein, Weyl, Niels Bohr, P. W. Bridgman and Wigner—who dig deep enough into the foundations of our knowledge to reach genuinely philosophical difficulties. The vast majority of working scientists, myself included, find comfort in the words of the French mathematician Henri Lebesgue: "In my opinion a mathematician, in so far as he is a mathematician, need not preoccupy himself with philosophy—an opinion, moreover, which has been expressed by many philosophers."

We are content to leave the philosophizing to giants such as Bohr and Wigner, while we amuse ourselves with the exploration of nature on a more superficial level. I shall accordingly not discuss further the ultimate reasons why mathematical concepts have come to be preeminent in physics. I shall beg the philosophical question, assuming as an article of faith that nature is to be understood in mathematical terms. The questions I shall address are practical ones relating to the way in which mathematical ideas react on physics. What are the standards of taste and judgment that mathematics imposes on the physicist? Which are the parts of mathematics that now offer hope for new physical understanding? In conclusion, since one concrete example is better than a mountain of prose, I shall sketch the role group theory has played in physics, leading up to the theory of fundamental particles known as the "eightfold way" [see "Strongly Interacting Particles," by Geoffrey F. Chew, Murray Gell-Mann and Arthur H. Rosenfeld; SCIENTIFIC AMERICAN, February]. This theory, developed independently by Gell-Mann and Yuval Ne'eman, has been brilliantly vindicated by the discovery of the omega-minus particle.

Before plunging into the details of

RATIO OF ORBITS	COPERNICAN VALUES	KEPLER'S MODEL	MODERN VALUES
MERCURY MAXIMUM / VENUS MINIMUM	.723	.707	.650
VENUS MAXIMUM / EARTH MINIMUM	.794	.795	.741
EARTH MAXIMUM / MARS MINIMUM	.757	.795	.735
MARS MAXIMUM / JUPITER MINIMUM	.333	.333	.337
JUPITER MAXIMUM / SATURN MINIMUM	.635	.577	.604

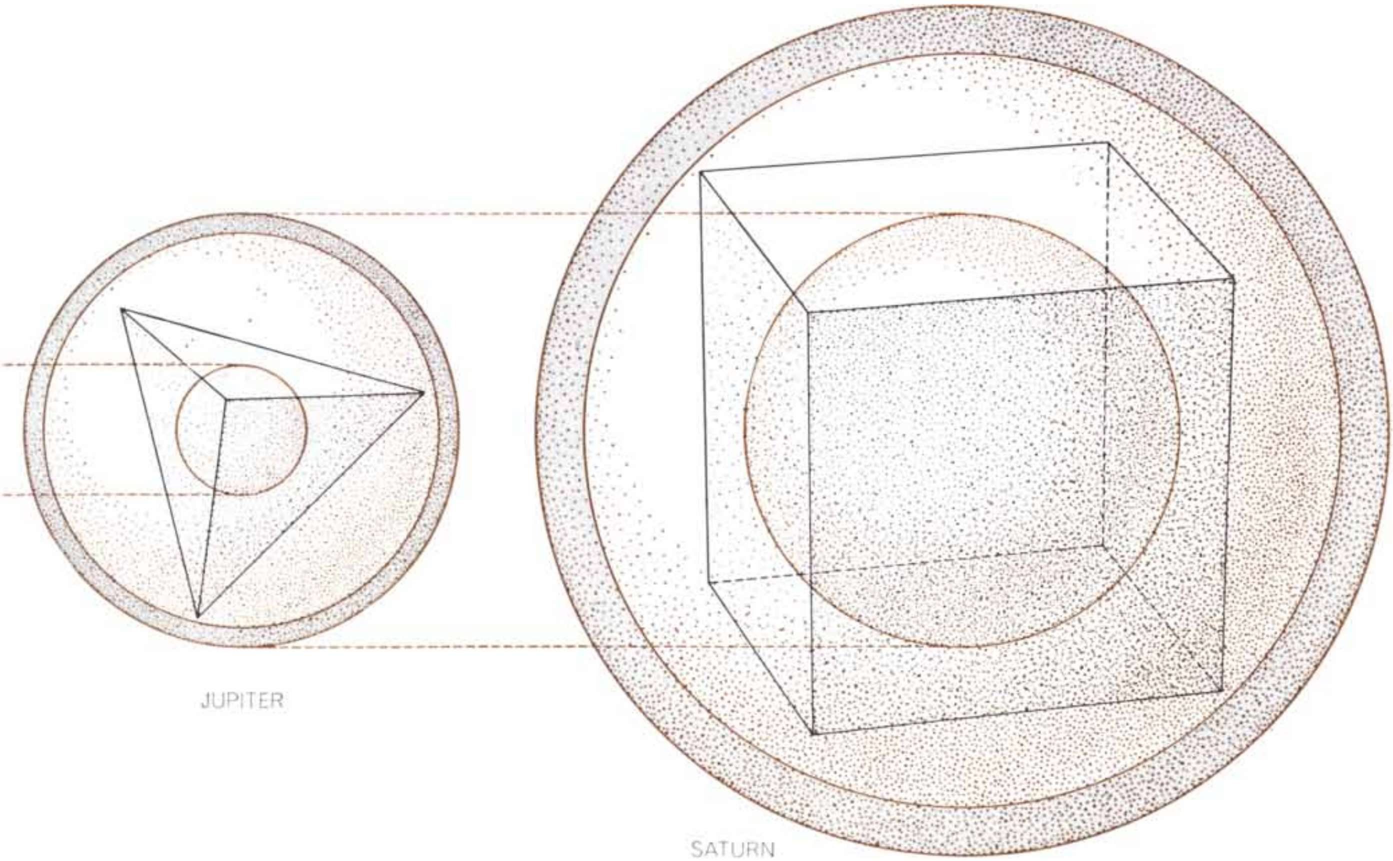
EXPLODED VIEW of Kepler's polyhedron model of the solar system (*right*) shows how each planetary orbit was supposed to occupy a spherical shell whose thickness corresponded to the difference between that planet's maximum and minimum distance from the sun. The table at left contains three sets of values for the ratio between each planet's maximum orbit and the next outer planet's minimum orbit. The first column gives the observational values obtained by Kepler from Copernicus. The second column gives the

present-day problems, I shall illustrate the effects of mathematical tastes and prejudices on physics with some historical examples. In trying to explain technical matters to a nontechnical audience, it is often helpful to examine past history and draw analogies between the problems of the past and those of the present. The reader should be warned not to take historical analogies too seriously. Very few active scientists are particularly well informed about the history of science, and almost none are directly guided in their work by historical analogies. In this respect scientists can be compared to politicians. The greatest politician of our century was probably Lenin, and he operated successfully within a historical viewpoint that was grossly limited and distorted. The only important historian of modern times to achieve high political office was François Guizot, prime minister of France during the 1840's, and all his historical understanding did not save him from mediocrity as a statesman. A good historian is too much committed to the past to be either a creative political leader or a creative scientist. In science at least, if a man wishes to achieve greatness, he should follow the advice of William Blake: "Drive your cart and your plow over the bones of the dead."

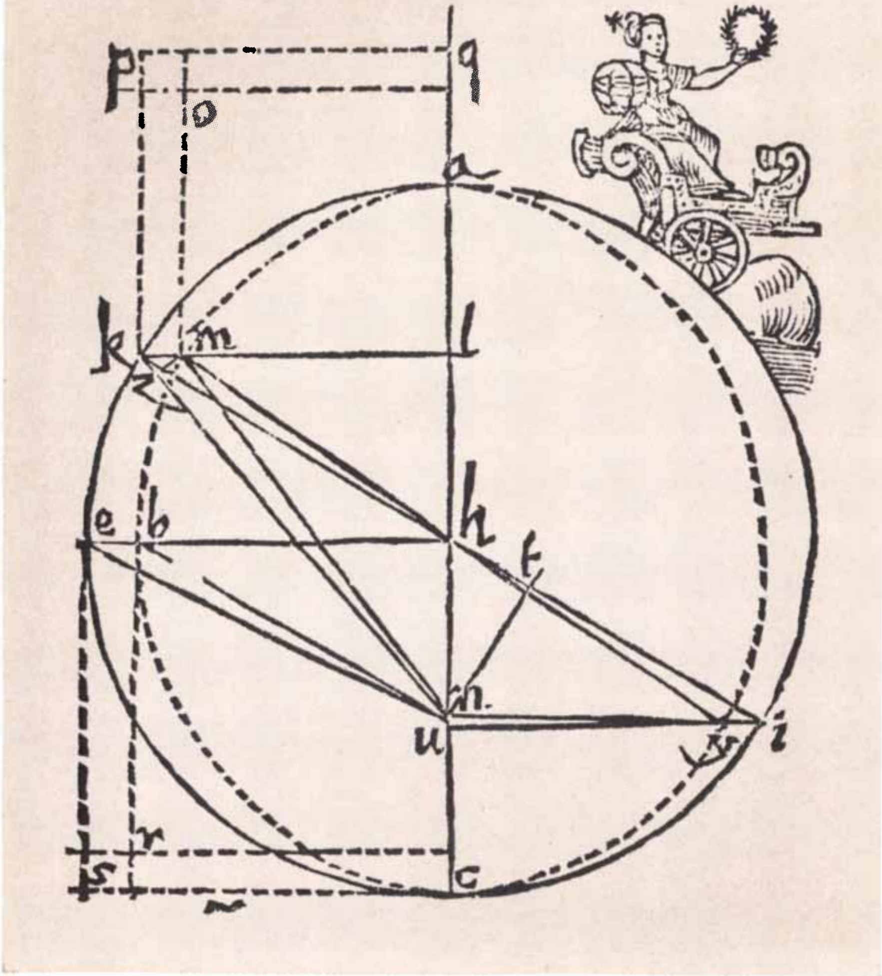
The most spectacular example in physics of the successful use of mathematical imagination is still Einstein's theory of gravitation, otherwise known as the general theory of relativity. To build his theory Einstein used as his working material non-Euclidean geometry, a theory of curved spaces that had been invented during the 19th century. Einstein took the revolutionary step of identifying our physical space-time with a curved non-Euclidean space, so that the laws of physics became propositions in a geometry radically different from the classical flat-space geometry [see "Geometry," page 60]. All this was done by Einstein on the basis of very general arguments and aesthetic judgments. The observational tests of the theory were made only after it was essentially complete, and they did not play any part in the creative process. Einstein himself seems to have trusted his mathematical intuition so firmly that he had no feeling of nervousness about the outcome of the observations. The positive results of the observations were, of course, decisive in convincing other physicists that he was right.

General relativity is the prime example of a physical theory built on a mathematical "leap in the dark." It might have remained undiscovered for a century if a man with Einstein's peculiar imagination had not lived. The same cannot be said of quantum mechanics, the other major achievement of 20th-century physics. Quantum mechanics was created independently by Werner Heisenberg and Erwin Schrödinger, working from quite different points of view, and its completion was a cooperative enterprise of many hands. Nevertheless, in quantum mechanics too the decisive step was a speculative jump of mathematical imagination, seen most clearly in the work of Schrödinger.

Schrödinger's work rested on a formal mathematical similarity between the theory of light rays and the theory of particle orbits, a similarity discovered some 90 years earlier by the Irish mathematician William Rowan Hamilton. Schrödinger observed that the theory of light rays is a special limiting case of



theoretical values predicted by Kepler's polyhedron model. The third column gives the accepted modern values. Kepler cheated in the case of Mercury in order to account for the most conspicuous discrepancy between his theory and the Copernican values: although the four outer polyhedra are circumscribed around a planetary shell in the usual way (the shell touches the faces of the polyhedron), the octahedron is circumscribed around the shell of Mercury in a special way (the shell touches the edges of the octahedron).



DISCOVERY OF ELLIPTICAL ORBIT FOR MARS was Kepler's great triumph after years of trying to make circular orbits satisfy Tycho Brahe's observations. In this diagram he shows that Mars sweeps out equal areas, measured from the sun at *n*, in equal times.

the theory of light waves that had been established after Hamilton's time by Maxwell and Hertz. So Schrödinger argued: Why should there not be a theory of particle waves having the same relation to particle orbits as light waves have to light rays? This purely mathematical argument led him to construct the theory of particle waves, which is now called quantum mechanics. The theory was promptly checked against the experimentally known facts concerning the behavior of atoms, and the agreement was even more impressive than in the case of the general theory of relativity. As often happens in physics, a theory that had been based on some general mathematical arguments combined with a few experimental facts turned out to predict innumerable further experimental results with unfailing and uncanny accuracy.

General relativity and quantum mechanics are success stories, showing mathematical intuition in a fruitful and liberating role. Unfortunately there is another side to the picture. Mathematical intuition is more often conservative than revolutionary, more often hampering than liberating. The worst of all the historic setbacks of physical science was the definitive adoption by Aristotle and Ptolemy of an earth-centered astronomy in which all heavenly bodies were supposed to move on spheres and circles. The Aristotelian astronomy benighted science almost completely for 1,800 years (250 B.C. to A.D. 1550). There were of course many reasons for this prolonged stagnation, but it must be admitted that the primary reason for the popularity of Aristotle's astronomy was a misguided mathematical intuition that held only spheres and circles to be aesthetically satisfactory.

Ptolemy explained the motions of the moon and planets by means of cycles and epicycles, that is to say, hierarchies of circles of various sizes moving one on another. The devastating feature of Ptolemy's system was that, since it was tailored in detail to fit the observed motions of each planet, it was immune to observational disproof. By the time of Ptolemy (A.D. 150) the vital force of Greek mathematics had been extinguished, and there were no new mathematical ideas to contest the stranglehold of Euclid's spheres and circles on the scientific imagination. Disturbed neither by new celestial observations nor by new mathematics, the 1,000-year night set in.

When Kepler in 1604 finally demolished the epicyclic cosmology by his discovery that planetary orbits are ellipses, he was not helped by any mathematical preconceptions favoring elliptical motions. On the contrary, he had to fight tooth and nail against his own mathematical prejudices, which were still uncompromisingly medieval. Only after years of struggling with various systems of epicycles did he overcome his conservative tastes enough to consider a system of ellipses. Such mathematical conservatism is the rule rather than the exception among the great minds of physics. The man who breaks out into a new era of thought is usually himself still a prisoner of the old. Even Isaac Newton, who invented the calculus as a mathematical vehicle for his epoch-making discoveries in physics and astronomy, preferred to express himself in archaic geometrical terms. His *Principia Mathematica* is written throughout in the language of classical Greek geometry. His assistant Henry Pemberton, who edited the third edition of the *Principia*, reports that Newton always expressed great admiration for the geometers of ancient Greece and censured himself for not following them more closely than he did. Lord Keynes, the economist, who made a hobby of collecting and studying Newton's unpublished manuscripts, summed up his impressions of Newton in the following words:

"In the eighteenth century and since, Newton came to be thought of as the first and greatest of the modern age of scientists, a rationalist, one who taught us to think on the lines of cold and untinctured reason. I do not see him in this light. I do not think any one who has pored over the contents of the box which he packed up when he finally left Cambridge in 1696 and which, though partly dispersed, have come down to us, can see him like that. Newton was not the first of the age of reason. He was the last of the magicians, the last of the Babylonians and Sumerians, the last great mind which looked out on the visible and intellectual world with the same eyes as those who began to build our intellectual inheritance rather less than 10,000 years ago. Isaac Newton, a posthumous child born on Christmas Day, 1642, was the last wonder-child to whom the Magi could do sincere and appropriate homage."

The character of Newton and his devotion to alchemy and to ancient apocalyptic writings are a fascinating subject, but it does not concern us here. We are concerned only with his mathematical style and tastes and with the effect of his mathematics on his science. Everything we know concerning his attitude toward mathematics is consistent with Keynes's conclusions. There is little doubt that Newton, like Kepler, made his discoveries by overcoming deeply conservative mathematical prejudices.

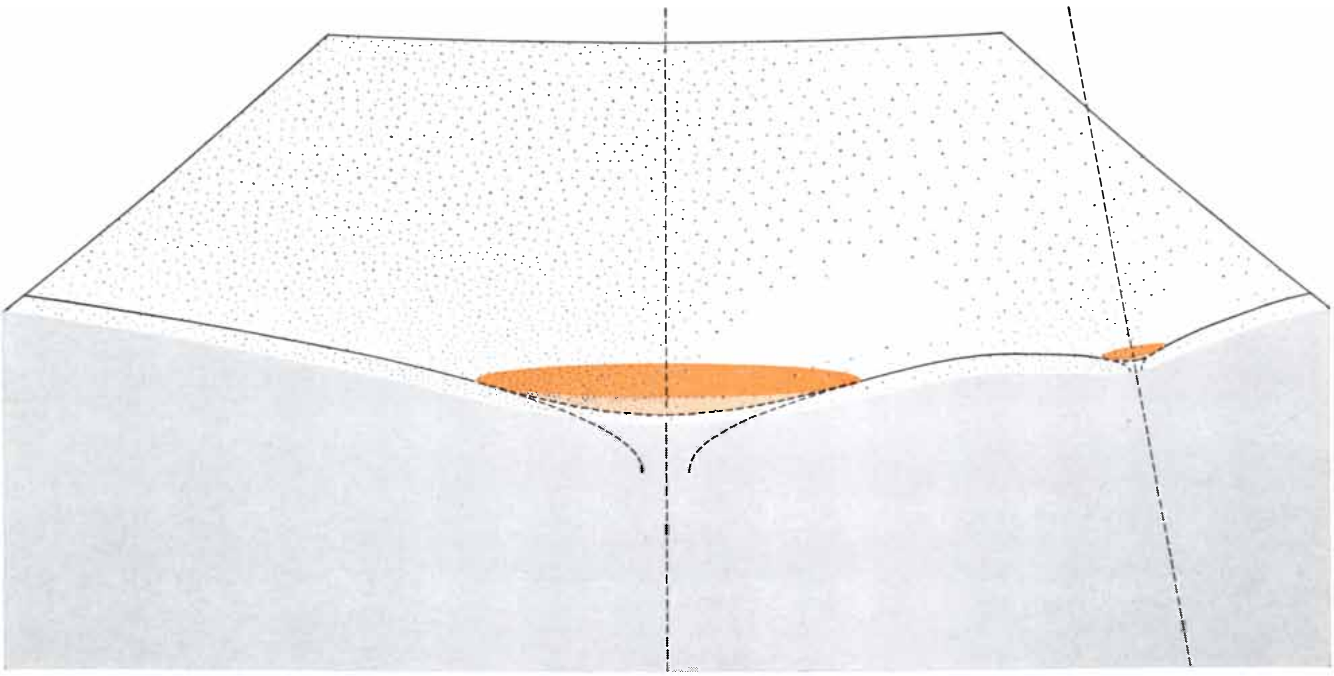
From these various historical examples we can only conclude that mathematical intuition is both good and bad, both indispensable to creative work in physics and also totally untrustworthy. The reasons for this two-edged quality lie in the nature of mathematics itself. As the physicist Ernst Mach remarked: "The power of mathematics rests on its evasion of all unnecessary thought and on its wonderful saving of mental operations." A physicist builds theories with mathematical materials, because the mathematics enables him to imagine more than he can clearly think. The physicist's art is to choose his materials and build with them an image of nature, knowing only vaguely and intuitively rather than rationally whether or not the materials are appropriate to his purpose. After the design of the theory is complete, rational criticism and experimental test will show if it is scientifically sound. In the process of theory-building, mathematical intuition is indispensable because the "evasion of unnecessary thought" gives freedom to the imagination; mathematical intuition is dangerous, because many situations in science demand for their understanding not the evasion of thought but thought.

I come now to discuss the present situation in physics. Without intending discourtesy to the experts in solid-state, nuclear spectroscopy and so forth, I use the word "physics" as an abbreviation for high-energy physics: the study of the fundamental particles. Physics (in this narrow sense) is now in an unusually happy situation. The latest generation of big accelerators has revealed during the past five years a whole new world of particles, with a quantity of detail and a richness of structure hardly anyone had expected. We must be profoundly thankful that the responsible physicists and politicians, not knowing that all these things were there, had the faith and courage 10 years ago to go ahead with building the machines. As a result of their enterprise we now have a large amount of exact information about a world that is as new and strange as the world of atoms was in 1910. Just as in 1910, we have no comprehensive theory, and the theorists have complete freedom to make of the experimental data what they will.

In this situation the theoretical physicists choose their objectives and their methods according to criteria of mathematical taste. The primary question for a theorist is not yet "Will my theory work?" but rather "Is what I am doing a theory?" The material at hand for theoretical work consists of fragments of mathematics, cookbook rules of calculation and a few general principles surviving from earlier days. What combination of these items would deserve the name of a theory is a question of mathematical taste.

The three main methods of work in contemporary theory are called field theory, S-matrix theory and group theory. They are not mutually exclusive; at least there is no contradiction between the things that adherents of the different methods do, although there is sometimes a contradiction between the things they say. Probably all three points of view will in the end make fruitful contributions to the understanding of nature.

The three methods differ not only in their choice of mathematical material but also in the uses to which the ma-

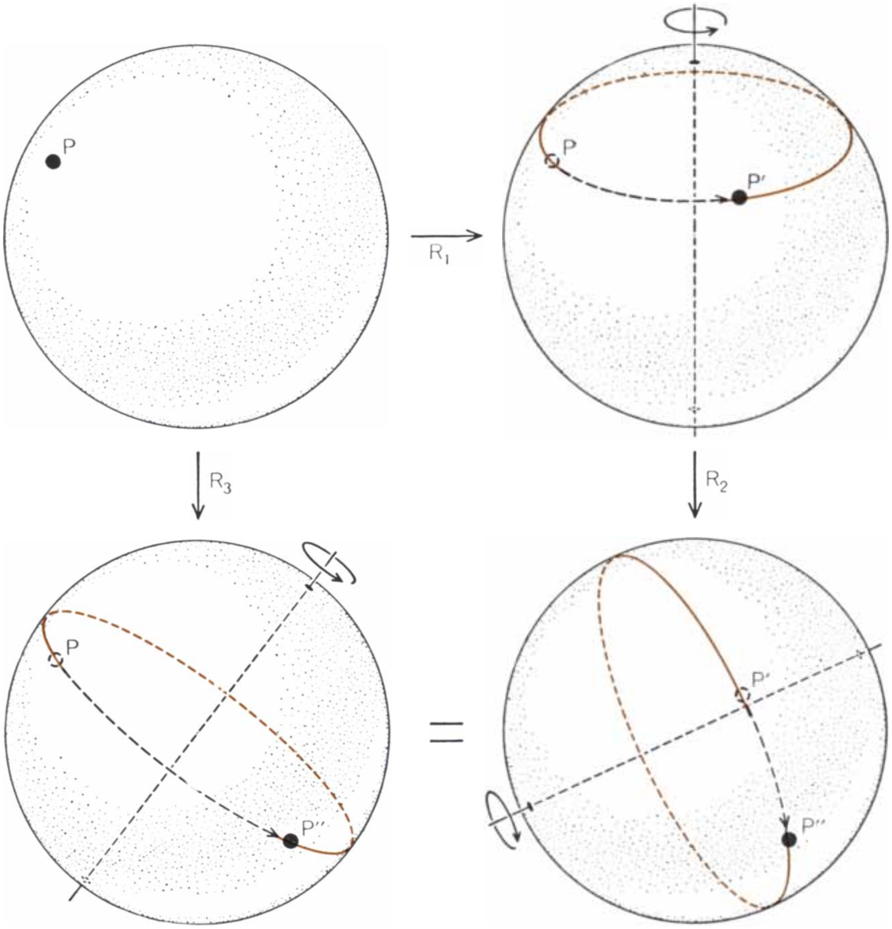


CURVATURE OF SPACE was postulated by Einstein on the basis of very general arguments and aesthetic judgments. To build his theory Einstein used as his working material non-Euclidean geometry, a theory of curved spaces that had been invented during the 19th century. In this representation two massive bodies are shown in two dimensions on a two-dimensional surface. The local curvature of space around the bodies accounts for their gravitational properties. In actuality physical space-time is four-dimensional.

terial is put. Field theory begins from a prejudice in favor of mathematical depth, a feeling that deep physical understanding and deep mathematics ought to go together. So the chosen mathematical material is the algebra of operators in Hilbert space, which is combined with various other difficult parts of mathematics in order to reach a structure that embodies some of the salient features of the real world. The emphasis is on a rigorous mathematical understanding of the theory, not on detailed comparison with experiment. Of the three methods, field theory is the remotest from experiment and the most mathematically strict, the most ambitious in its intellectual tone and the vaguest in its relevance to physics. I am myself addicted to it and am therefore particularly qualified to point out its limitations.

In *S*-matrix theory (the *S* stands for *Streu*, the German word for "scatter") the mathematical material is deliberately chosen to be as elementary as possible. It consists of the standard theory of analytic functions of complex variables, a theory whose essential features have not changed since it was created by the French mathematician Augustin Cauchy early in the 19th century. *S*-matrix theory compensates for the weakness of its mathematical base by making heavy use of experimental data. The *S*-matrix theoretician typically aims to compute or predict the result of one experiment by making use of the results of others. Sometimes predictions are made from "first principles" independent of other experiments, and the hope is ultimately to deduce everything from first principles. One of the most pleasant and refreshing features of *S*-matrix theory is that the rules of the game can be changed as a calculation proceeds. The method as it now exists is transitional; one is not applying a cut-and-dried theory but rather creating a theory as one goes along by a process of trial and error. At every stage of the work the comparison with experiment will ruthlessly eliminate the unfit idea and leave room for truth to grow.

The success of *S*-matrix theory in interpreting experiments, and in giving guidance to experimenters, has been impressive. My own preference for field theory is based on a personal taste that, judged by the evidence of history, cannot be considered reliable. I find *S*-matrix theory too simple, too lacking in mathematical depth, and I cannot believe that it is really all there is. If the *S*-matrix theory turned out to explain everything, then I would feel disappointed that the Creator had after all been rather unsophisticated. I realize, however, that He has a habit of being sophisticated in ways one does not expect.



THREE-DIMENSIONAL ROTATION GROUP O_3 is defined as the set of all rotations of an ordinary three-dimensional space about a fixed center. If R_1 and R_2 are any two such rotations, the result of combining the two can be duplicated by a third rotation, R_3.

I shall now discuss group theory, the third of the principal methods used in modern theoretical physics, in somewhat greater detail than the other two. The mathematical material here is a theory of considerable depth and power, mostly dating from the first quarter of the 20th century. The two main concepts are "group" and "representation." A group is a set of operations possessing the property that any two of them performed in succession are together equivalent to another operation belonging to the set. For example, the three-dimensional rotation group O_3 is defined as the set of all rotations of an ordinary three-dimensional space about a fixed center. Obviously, if R_1 and R_2 are any two such rotations, the combination of R_1 with R_2 can be duplicated by a third rotation, R_3. A representation of a group is a set of numbers and a rule of transformation of these numbers such that each operation of the group produces a well-defined transformation of the numbers. The transformations in a representation are restricted to being linear; that is to say, if a particular transformation sends p to p' and q to q', then it also sends $p + q$ to $p' + q'$. An example of a representation of O_3 is the set of three coordinates (x, y, z) that determine the position in space of any point P [*see illustration at left*]. When a rotation R is applied, the point P moves to a new position P' with coordinates x', y', z', and this determines the rule of transformation for x, y, z. This particular representation of O_3 is called the triplet representation, since there are three numbers involved in it.

The immense power of group theory in physics derives from two facts. First, the laws of quantum mechanics decree that whenever a physical object has a symmetry, there is a well-defined group (G) of operations that preserve the symmetry, and the possible quantum states of the object are then in exact correspondence with the representations of G. Second, the enumeration and

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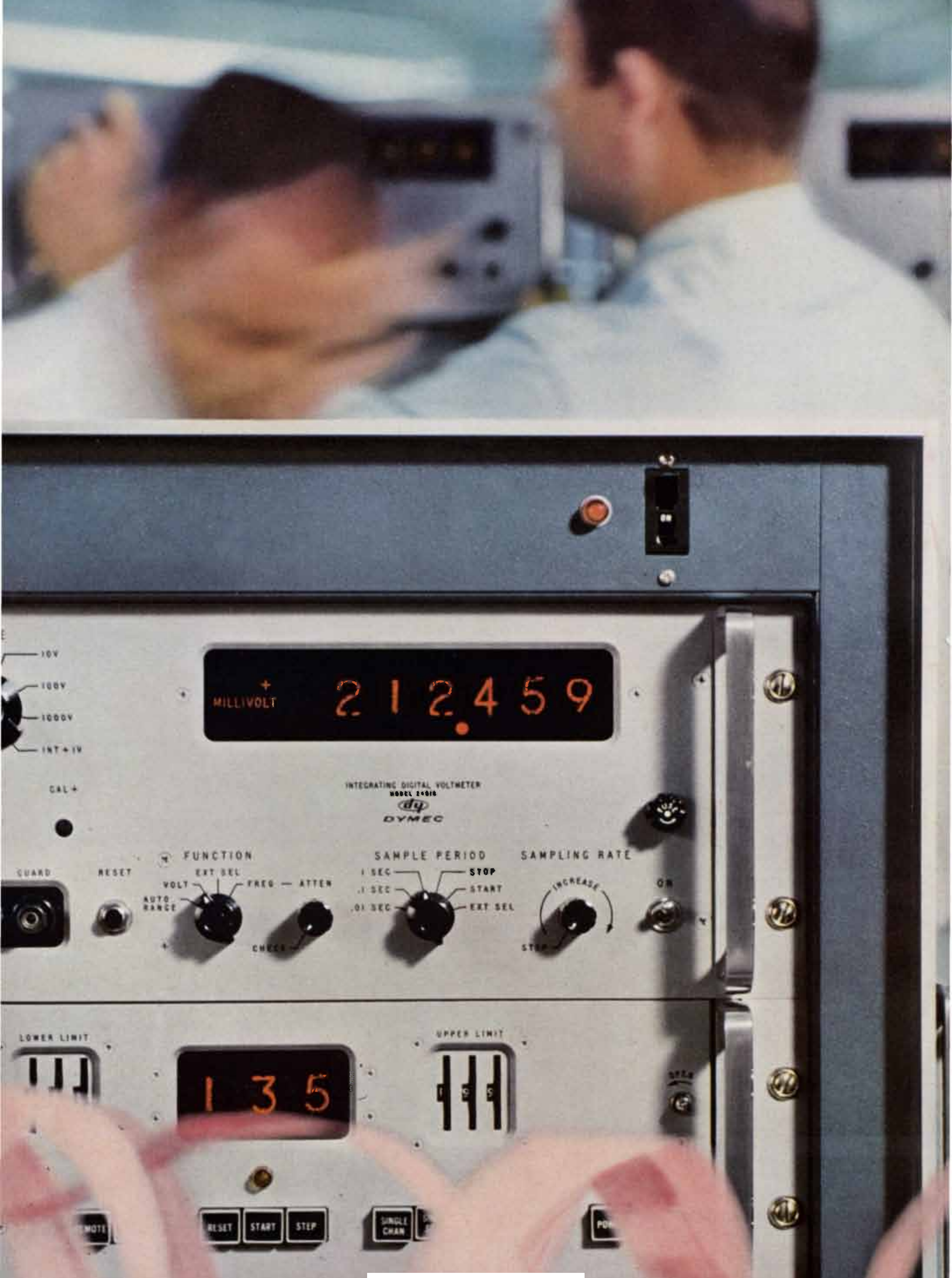
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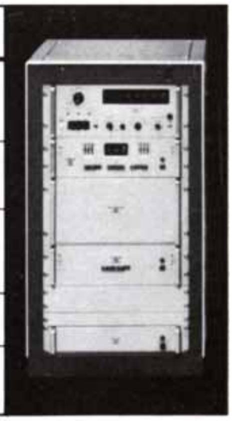

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MASTER NAVIGATORS THROUGH TIME AND SPACE

GUIDANCE AND NAVIGATION FOR SPACECRAFT • BALLISTIC MISSILES • AVIONICS • SPACE BOOSTERS • DELIVERED ON TIME AT LOW COST WITH OUTSTANDING ACCURACY AND RELIABILITY classification of all well-behaved groups and of their representations have been done by the mathematicians, once and for all, independently of the physical situation to which the groups may be applied. From these two facts results the possibility of making a purely abstract theory of the symmetries of fundamental particles, based on the abstract qualities of groups and representations and avoiding all arbitrary mechanical or dynamical models.

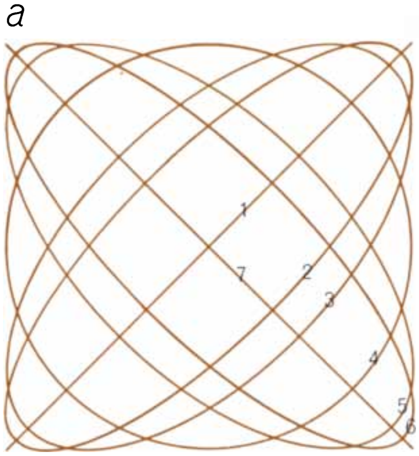
The crucial transition from concrete to abstract group theory is most easily explained by examples. An atom floating in a rarefied gas has no preferred direction in space and therefore has the symmetry of the ordinary rotation group O_3. Among the representations of O_3 there is the triplet representation. Those states of the atom that have one unit of spin belong to this representation and are called triplet states; they always occur precisely in groups of three with the same energy. Now let a magnetic field be turned on so as to destroy the rotational symmetry; the three equal energies are slightly split apart and the three states can be seen in a spectroscope as a visible triplet of spectral lines. Such a classification of states of the atom according to their rotational symmetry is the standard example of concrete group theory at work.

Now we jump to a different example. There are three kinds of fundamental particle called pions, one positively charged, one negatively charged and one neutral. They all have approximately the same mass and approximately the same nuclear interactions. Let us then imagine that they are a triplet representation of a group O_3', having exactly the same abstract structure as O_3 but having nothing to do with ordinary space rotations. We can then predict many of the properties of pions from abstract group theory alone without knowing anything about the intrinsic nature of the operations constituting O_3'. It turns out that all of these predicted properties of pions are correct. What is much more remarkable, these predictions were made on the basis of abstract group theory by Nicholas Kemmer in 1938, nine years before the first pion was discovered. The group O_3' (with some slight modification) is known in physics as the "isotopic-spin group."

Finally we come to the eightfold way, which gave us the key to the classification of the more recently discovered particles. The classification depends on a group U_3, which is larger and less familiar than O_3. To make U_3 understandable to nonmathematicians I shall introduce a mechanical model that bears the same relation to the abstract group U_3 as the rotations in three-dimensional space bear to the abstract group O_3. Needless to say, this mechanical model is not supposed to exist in the real world. It is intended only to illustrate the structure of U_3.

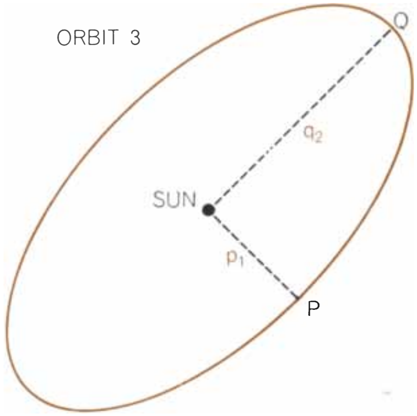
Consider a solar system in which the force of gravity varies directly with the first power of distance instead of with the inverse-square law. Suppose the planets to be small, so that their mutual perturbations are negligible. Each planet then moves independently in an elliptical orbit with the sun at the center. The peculiar feature of these orbits is that they all have the same period, the outer planets moving faster than the inner ones. We call the period of each orbit a "year," so that the positions of

a



ORBIT	1	2	3	4	5	6	7
p_1	0	1	$\sqrt{2}$	$\sqrt{5}$	$\sqrt{8}$	3	$\sqrt{10}$
q_2	$\sqrt{10}$	3	$\sqrt{8}$	$\sqrt{5}$	$\sqrt{2}$	1	0

b



ORBIT 3: $p_1 = \sqrt{2}$, $q_2 = \sqrt{8}$

$p_2 = q_1 = p_3 = q_3 = 0$

TOTAL ENERGY OF ORBIT 3 $= [p_1^2 + q_2^2] = [2 + 8] = 10$

c

REPRESENTATION OF ORBITS IN EIGHTFOLD WAY

$a_{12} = p_1q_2 - p_2q_1$

$a_{23} = p_2q_3 - p_3q_2 = 0$

$a_{31} = p_3q_1 - p_1q_3 = 0$

$S_{11} = p_1^2 + q_1^2$

$S_{22} = p_2^2 + q_2^2$

$S_{33} = p_3^2 + q_3^2 = 0$

$S_{12} = p_1p_2 + q_1q_2 = 0$

$S_{23} = p_2p_3 + q_2q_3 = 0$

$S_{31} = p_3p_1 + q_3q_1 = 0$

d

VALUES OF S_{11}, a_{12}, S_{22}

ORBIT	1	2	3	4	5	6	7
S_{11}	0	1	2	5	8	9	10
a_{12}	0	3	4	5	4	3	0
S_{22}	10	9	8	5	2	1	0

TOTAL ENERGY FOR ALL ORBITS $= S_{11} + S_{22} = 10$

EIGHTFOLD-WAY MODEL bears the same relation to the abstract group SU_3 as rotations in three-dimensional space (*illustrated on page 134*) bear to the abstract group O_3. The model (*a*) shows seven planetary orbits that can be transformed into each other by operations belonging to the group SU_3, discussed in the text. That there are seven orbits is not significant; any number of others could be specified to satisfy the needs of this particular model. Orbit No. 3 is shown separately (*b*) to indicate how a planetary motion is defined by the points *P* and *Q*, with values p_1 and q_2. Normally six coordinates (three of *p* and three of *q*) are needed to define a point in space. But because of the special way the coordinate axes are chosen for this model, p_2 and q_1 are zero, and because the orbits all lie in a plane, the coordinates p_3 and q_3 are also zero. All the orbits have the same total energy ($p_1^2 + q_2^2 = 10$) but different angular momenta, expressed in terms of the value a_{12}. In particular the two straight-line orbits (*1 and 7*) have zero angular momentum, whereas the circular orbit (*4*) has the most angular momentum. According to the eightfold way the seven orbits can be represented by sets of nine numbers, listed in *c*. It is evident that six of these numbers vanish because p_2, p_3, q_1 and q_3 are all zero. Thus only three components remain: S_{11}, a_{12}, S_{22}. When the appropriate values for p_1 and q_2 are inserted, the three components take the values shown in *d*. The values are such that they transform into each other when the operations of SU_3 symmetry are applied. This is possible, in part, because the total energy for all orbits is the same: $S_{11} + S_{22} = 10$.

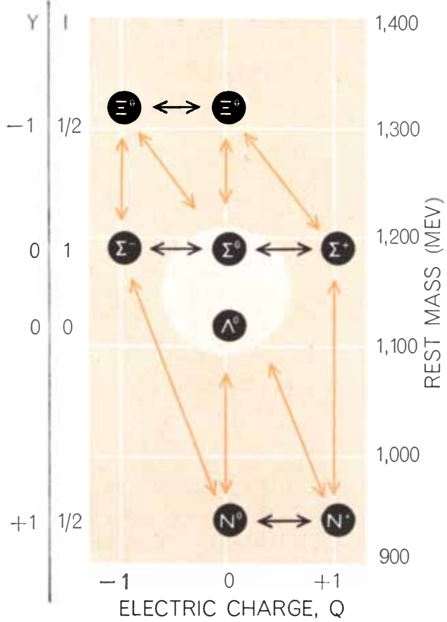
all planets repeat themselves at yearly intervals.

The motion of a planet can be specified precisely by two points in space denoted (P, Q), P being the position of the planet now and Q being the position it will occupy three months later. Another planet traveling three months ahead of the first in the same orbit will be specified by $(Q, -P)$, where $-P$ means the point diametrically opposite P. The total energy of either of these planets is given by $(OP^2 + OQ^2)$, which is the sum of the squares of the distances of the points P and Q from the sun at O. The group U_3 (as exhibited by this particular model) is defined as the set of all transformations of the planetary motions, subject to the following three restrictions: (1) the transformations are linear; (2) the transformations leave the total energy of each motion unchanged, and (3) if two or more planets are moving in a given orbit, a transformation that takes one to a new orbit takes all.

If only the first two conditions were imposed, we would have the group of all linear transformations of (P, Q), leaving the sum $(OP^2 + OQ^2)$ unchanged. This would be simply the rotation group O_6 in a space of six dimensions (three dimensions for P and three for Q). The group U_3 is thus a subgroup of O_6. The third restriction on U_3 can be stated in a more concise but equivalent form as follows: A transformation that takes the motion (P, Q) into (R, S) must also take $(Q, -P)$ into $(S, -R)$.

Two special kinds of transformation can easily be seen to belong to U_3. First, consider ordinary rotations operating on P and Q simultaneously. These obviously satisfy the three conditions. Hence the rotation group O_3 is a subgroup of U_3. Second, consider the transformations of time-displacement, in which each planet is transformed into the same planet at a fixed interval of time earlier or later. The time-displacements also belong to U_3 and form another subgroup (T) of U_3.

For application to physics it is convenient to reduce U_3 to a smaller group SU_3 (and when I write SU_3, I mean the group the professionals call SU_3/Z_3). One obtains SU_3 from U_3 by simply forgetting time. For SU_3 all motions belonging to the same orbit are regarded as identical, irrespective of the time. Whereas U_3 transforms a planetary motion into another planetary motion at a particular time, SU_3 transforms an orbit into an orbit without reference to time. In mathematical language SU_3 is the group U_3 with the subgroup T of time-displacements removed from it. The representations of SU_3 are precisely those representations of U_3 that are independent of time.



SUPERFAMILY OF EIGHT was the first grouping suggested by the eightfold way. It contains the eight most familiar baryons: the neutron (N^0) and proton (N^+)—also known as the nucleon doublet—the lambda (Λ) singlet, the original sigma (Σ) triplet and the original xi (Ξ) doublet. The sigma triplet and xi doublet that appear in the 10-member superfamily containing the omega minus (*see page 128*) are heavier particles with the same values of *Y* and *I*.

Let us now look for simple representations of SU_3. A planetary motion (P, Q) is defined by the six coordinates $(p_1, p_2, p_3, q_1, q_2, q_3)$ of P and Q. These coordinates by themselves define a representation of U_3 but, since they are time-dependent, not of SU_3. The simplest time-independent quantities (for reasons we need not go into) consist of p's and q's multiplied together in various combinations, as shown in the illustration on the preceding page. There are nine and only nine such quantities. Three of them are components of the angular momentum and are designated a_{12}, a_{23} and a_{31}; six are components of another kind that are related to the total energy of the system: S_{11}, S_{22}, S_{33}, S_{12}, S_{23}, S_{31}. (The subscripts indicate which of the p's and q's are involved in defining the quantity, thus $a_{12} = p_1q_2 - p_2q_1$ and $S_{12} = p_1p_2 + q_1q_2$.)

Because this representation of SU_3 involves nine quantities it is said to be nine-dimensional. The sum $S_{11} + S_{22} + S_{33}$, however, is the total energy of the system and does not transform at all

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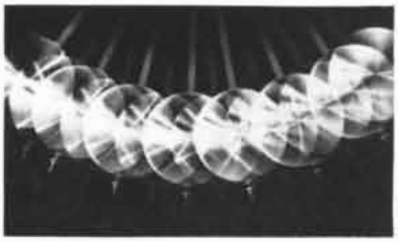

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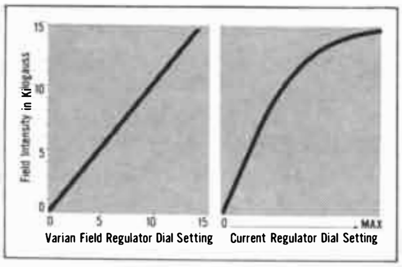

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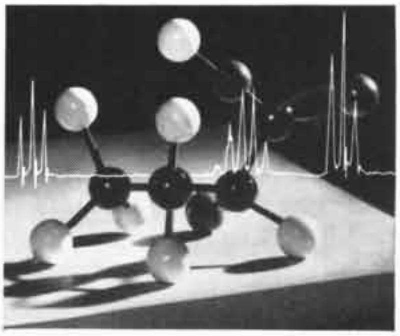

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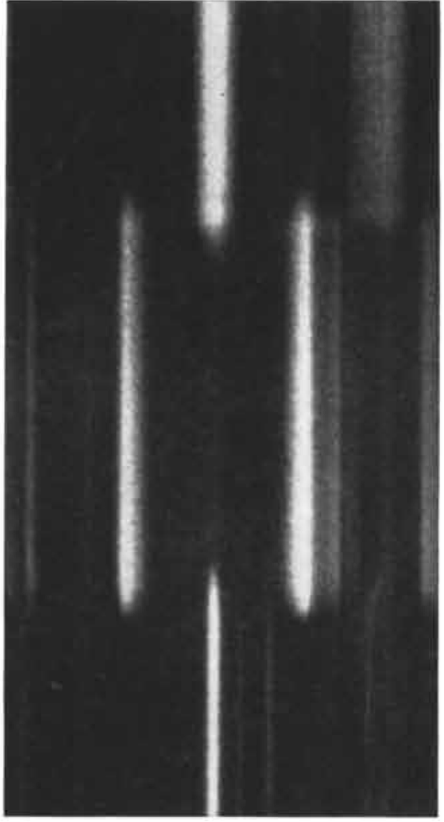
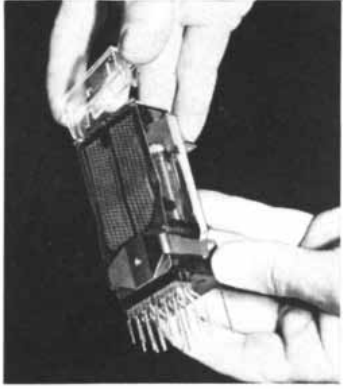
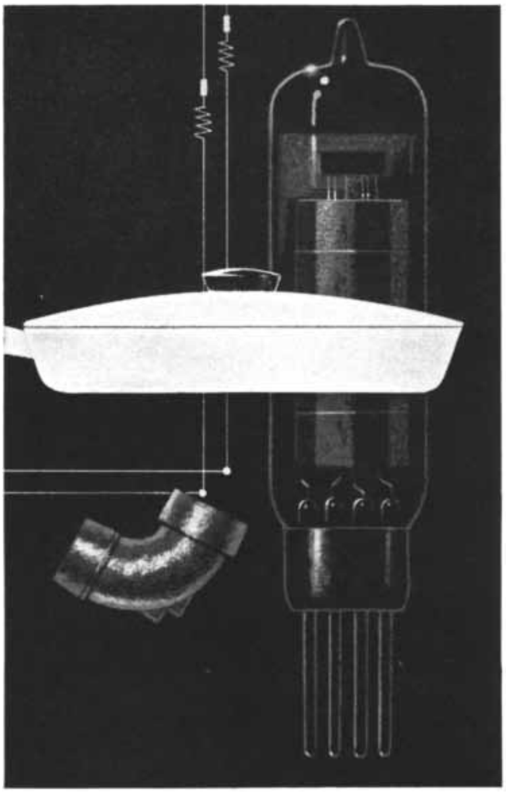
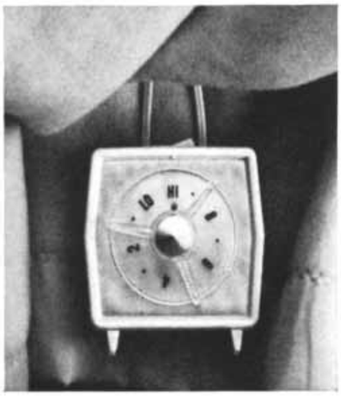


SPECTRAL LINE OF NIOBIUM (*bottom*) is split into three components when a magnetic field is turned on so as to destroy the rotational symmetry of the atoms. Two components are observed perpendicular to the magnetic field (*middle*) and a third is observed parallel to the magnetic field (*top*). The triplet lines correspond to three states of the atom that have one unit of spin each; these states always occur precisely in groups of three with the same energy. Such a classification of states of the atom according to their rotational symmetry is an example of applied group theory. Spectrograms were made in the Spectroscopy Laboratory of the Massachusetts Institute of Technology.

under any of the operations of U_3. When three quantities yield a constant sum, it is evident that only two of them are independent and that the third is always implied when any two are given. The three quantities S_{11}, S_{22} and S_{33} might be reduced to two independent quantities in various ways, but for technical reasons they are usually combined as follows: one quantity is expressed as $S_{11} - S_{22}$ and the other as $S_{33} - \frac{1}{2}(S_{11} + S_{22})$. As a result there are only five independent components of S rather than six, and these together with the three components of a yield a total of eight quantities that do in fact transform into each other under U_3. These eight quantities are time-independent and provide an eight-dimensional representation of SU_3. The representation







so obtained is the simplest that exists and is the famous eightfold way.

Finally, let us imagine that the SU_3 symmetry in nature is not perfect. Suppose, for example, that the "three direction" (that is, the direction in which the coordinates have the values p_3 and q_3) is somehow different from the other two directions. In terms of our imaginary solar system this means that symmetry is preserved only when the orbits all lie in the same plane and is not preserved by rotations that carry the orbits off the plane. In this case the symmetry group U_3 will be replaced by its subgroup U_2, consisting only of those transformations of U_3 that leave the three directions unaltered. Under the operations of U_2 the eightfold way does not remain a unified representation. Its eight components split into subsets in the following manner:

$S_{33} - \frac{1}{2}(S_{11} + S_{22})$ (a singlet)

$\left.\begin{matrix} S_{23},\ S_{31} \\ a_{23},\ a_{31} \end{matrix}\right\}$ (two doublets)

$(S_{11} - S_{22}),\ S_{12},\ a_{12}$ (a triplet)

Each of these subsets forms a representation of U_2. In other words, the transformation defined by the singlet representation pertains to a unique subset of the total set: a subset of one member. Each of the two doublets represents a slightly larger subset: a subset of two members. Similarly, the triplet subset has three members.

Turning now to the actual physical world, compare this eight-member structure with the eight original baryons, the most familiar of the heavy "elementary" particles, which consist of the lambda (Λ) singlet, the proton-neutron (or nucleon) doublet, the xi (Ξ) doublet and the sigma (Σ) triplet. The agreement is exact.

In other representations of SU_3 there are 10, 27 or more members. Gell-Mann was the first to point out that the symmetry of a 10-member set could be satisfied by nine of the known baryons if they were augmented by a missing singlet that he named, in advance, the omega-minus (Ω^-) baryon. The known members of this 10-member set were a delta (Δ) quartet, another sigma triplet and another xi doublet. The predicted singlet was discovered in February in bubble-chamber photographs made at the Brookhaven National Laboratory.

The evidence is now overwhelming that an abstract symmetry with the structure of SU_3 actually exists in nature and dominates the behavior of the

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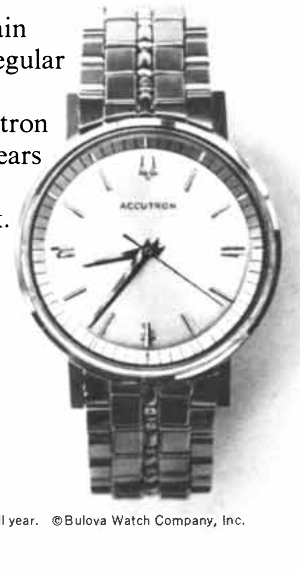

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strongly interacting particles. The symmetry is not perfect, being broken by some relatively weak perturbation that reduces the group SU_3 to its subgroup U_2. The U_2 symmetry that remains is essentially identical with the abstract isotopic-spin symmetry discussed earlier. Our entire picture of the strongly interacting particles has been transformed from chaos to a considerable degree of order by these compellingly simple group-theoretical ideas.

Group theory is in many respects the most satisfactory of the three theoretical methods I have discussed. Unlike S-matrix theory, it has an elegant and impeccably rigorous mathematical basis; unlike field theory, it has clear and solid experimental support. What then is lacking? The trouble with group theory is that it leaves so much unexplained that one would like to explain. It isolates in a beautiful way those aspects of nature that can be understood in terms of abstract symmetry alone. It does not offer much hope of explaining the messier facts of life, the numerical values of particle lifetimes and interaction strengths—the great bulk of quantitative experimental data that is now waiting for explanation. The process of abstraction seems to have been too drastic, so that many essential and concrete features of the real world have been left out of consideration. Altogether group theory succeeds just because its aims are modest. It does not try to explain everything, and it does not seem likely that it will grow into a complete or comprehensive theory of the physical world.

We are left with three methods of work in theoretical physics: field theory, S-matrix theory and group theory. None of them really deserves the name of theory, if we mean by a theory something similar to the great theories of the past, for example general relativity or quantum mechanics. They are too vague, too partial or too fragmentary. This is of course only my personal judgment. Even if they succeed in their declared aims, they do not satisfy my aesthetic sense of what a theory ought to be. I am tempted to apply to them the term "Bridges of snow built across crevasses of ignorance" to describe my feelings of dissatisfaction. This splendid phrase is often useful for characterizing theoretical ideas with which one happens to be unsympathetic. It is well to remember, however, that it was first so used by the bigoted biometrician Karl Pearson in a diatribe against Gregor Mendel's laws of inheritance.

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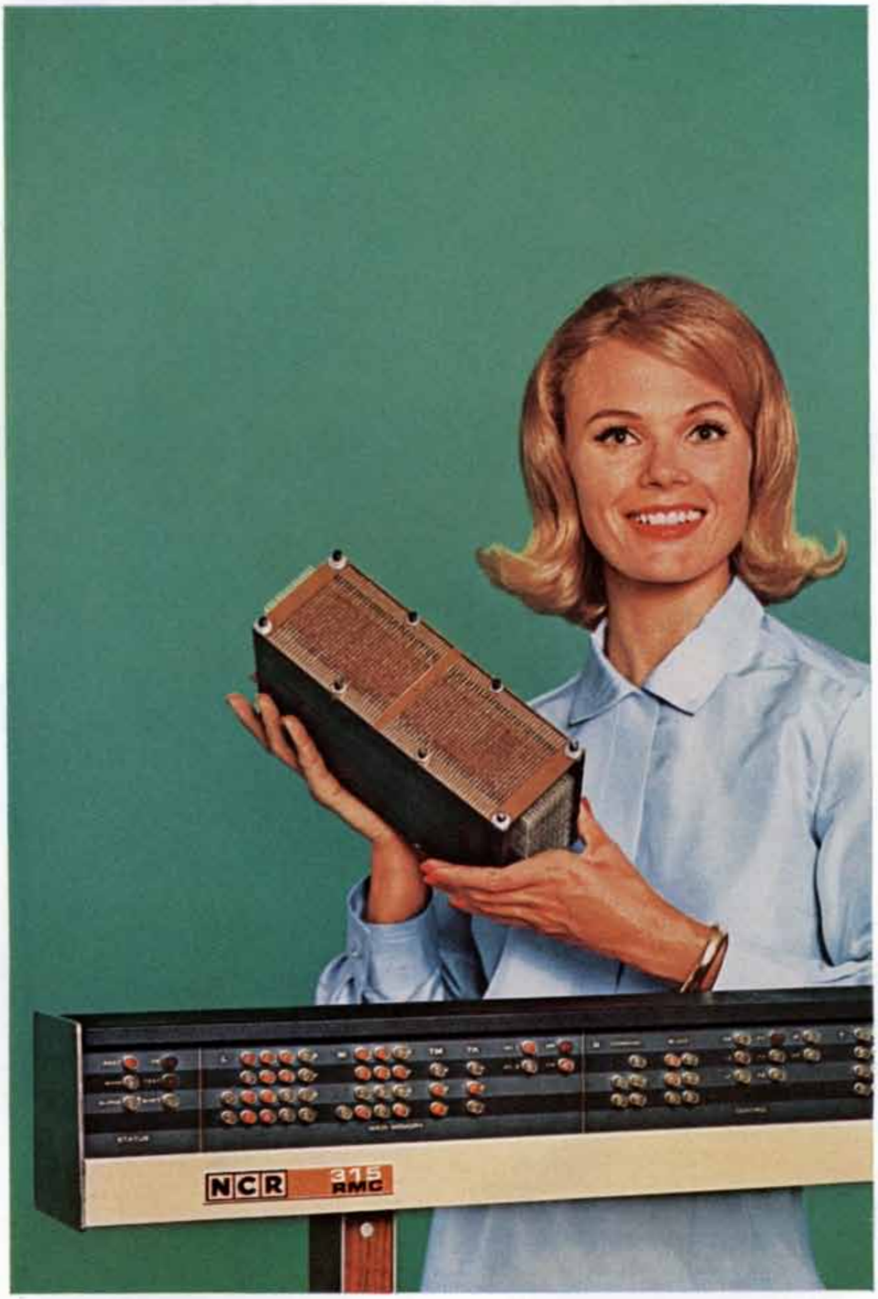

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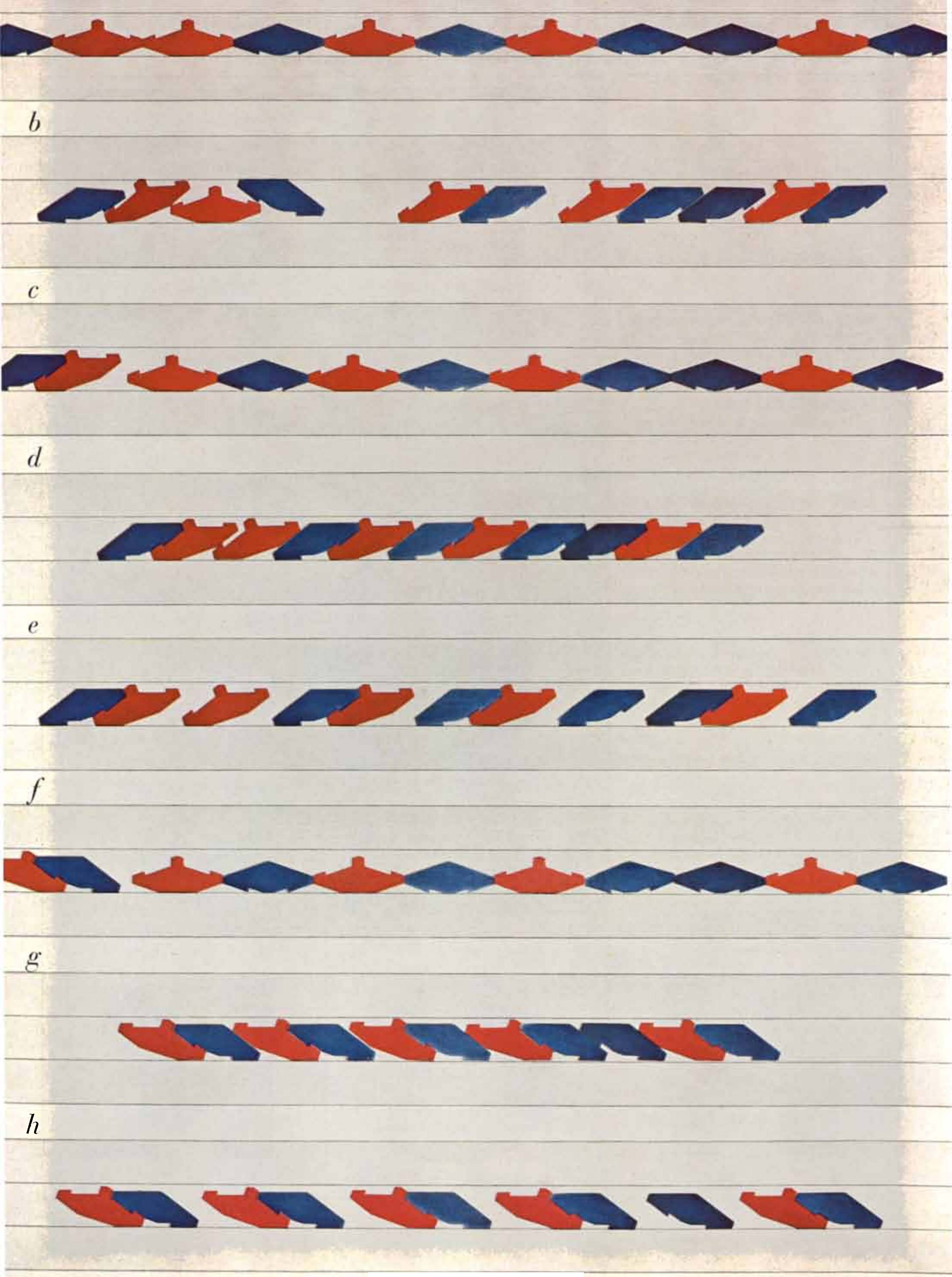
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Mathematics in the Biological Sciences

Biologists use mathematics, but the complex systems they study resist mathematical description. The kind of description that might someday be helpful is suggested by the abstract analysis of self-reproduction

by Edward F. Moore

René Descartes looked on animals (and the human body, but not the soul) as devices that could be explained in mechanical terms. Once when he was tutoring Queen Christina of Sweden, she challenged him with the question: But how can a machine reproduce itself? The same question is being asked today by mathematicians who reflect on the capabilities and limitations of machines. In their efforts to construct a detailed mathematical theory of self-reproduction they apply mathematical methods to the study of a process that has always seemed uniquely biological.

Such applications of mathematics have been rare. Whereas physics and mathematics have flowered together over the past 300 years, closely related to each other and mutually stimulating [see "Mathematics in the Physical Sciences," page 128], original mathematical thought has done little for biology. I can think of a few exceptions. Thomas Malthus created a mathematical model in which population multiplies in a geometric progression while the food supply increases only arithmetically, precipitating a "struggle for existence." Both Charles Darwin and Alfred Russel Wallace, on reading Malthus, saw in that struggle the mechanism with which to explain natural selection. An essentially mathematical idea, in other words, can be said to have contributed to the development of the central concept of biological evolution.

As for contributions by biology to mathematics, they too have been rare. One of the most notable has been in the area of population genetics. R. A. Fisher of Britain and Sewall Wright of the U.S. developed mathematical models to show how the rules of heredity combine with the laws of chance to cause a given gene to survive or die out in a population. Wright's extremely sophisticated model was based on diffusion theory, and it led William Feller of Princeton University to explore new areas of mathematics.

In their everyday work, of course, biologists do make use of mathematics. Like other investigators, biologists must subject their results to statistical tests (some of them developed by Fisher), and they customarily display the relations they discover in the form of the curves of analytic geometry. The mathematical equations of thermodynamics are familiar to biochemists. Statistical techniques have played a role in the deciphering of the genetic code and in the mapping of genes. The fact remains that all this is conventional mathematics. Although there have been a number of deliberate attempts to establish a "mathematical biology," most of these efforts seem not to fulfill their early promise. It is possible, however, that new mathematical studies utilizing computers may accomplish more than could be done in the past with necessarily oversimplified models of biological processes.

The special value of mathematics to biology lies not in its use as a tool but in its power to abstract and thus to lay bare fundamental problems and the relations between superficially distinct entities and processes. Organisms are machines, albeit highly organized machines. It seems to me that the significant encounter between mathematics and biology will stem from the logical examination of problems in machine theory that turn out to have relevance to fundamental problems in biology, and it is to speculations of this kind that this article will be largely confined.

Claude E. Shannon of the Massachusetts Institute of Technology and John McCarthy of Stanford University once pointed out that when men find machine analogues for the human body, the machines they pick necessarily reflect their times. Descartes compared the body to intricate water clocks and fountains; earlier in this century the brain was regarded as a telephone central office; more recently the electronic computer has been the preferred machine to which to compare an organism. Perhaps for this reason most of the workers who have recently explored the relations between organism and machine have concentrated on the central nervous system. They have tended to ask two questions: Can one explain the brain as a kind of computer? Can one

SELF-REPRODUCTION is one of the biological processes that have been subjected to mathematical analysis. The little red and blue "creatures" in the photographs on the opposite page are the two kinds of part of an elementary self-reproducing machine designed by the geneticist L. S. Penrose. If the parts are placed in a plastic tray (*a*) and the tray is shaken end to end, the parts are jostled but they do not link up (*b*). But if a blue-red "seed," or "machine," is introduced (*c*) and the parts are shaken again, the seed imparts a tilt to the other parts, which link to form more blue-red units (*d*); these machines are spread apart for better visibility in *e*. Now if a red-blue seed is introduced instead (*f*), shaking produces only red-blue machines (*g*, *h*): the machine "breeds true." Penrose's creatures were made of plywood; these were made of aluminum at the Bell Telephone Laboratories.

build a computer that "thinks" like a brain?

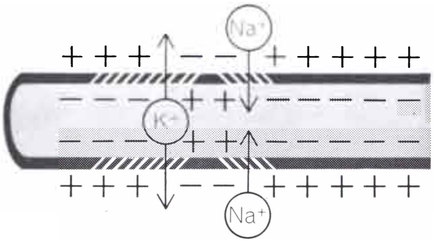
There have been several lines of approach, but central to most of them has been the concept of the "automaton," which is in effect an idealized machine or machine part or, when the analogy to neurophysiological processes is being pressed, an idealized organism or part of an organism, such as a cell. In automaton theory one deals not with the internal workings of the automaton but with its external manifestations. In the words of the late John von Neumann, the elements of a machine or an organism "are viewed as automatisms, the inner structure of which need not be disclosed, but which are assumed to react to certain unambiguously defined stimuli, by certain unambiguously defined responses."

One of the most useful of such abstract machines is the "finite-state machine," or "finite automaton." It is a "black box" that has a finite number of discrete internal "states." It also commonly has a finite number of possible inputs and possible outputs. The state and the output at any time T depend in some arbitrary way on the previous state and input at time $T-1$. Confronted with a finite automaton and a set of rules for its transitions from state to state, one can determine from its starting state and the input sequence what the state and output of the automaton will be at any specified time. The rules can be presented in the form of a table or diagram [*see top illustration on page 152*]. It is important in dealing with finite-state machines to understand the concept of state. The state of an automaton that is an abstract model of a combination lock is not "locked" or "open," for example; it is the situation of the lock's innards—the positions of the various unseen gears and levers, which change with each twist of the visible dial [*see bottom illustration on page 152*].

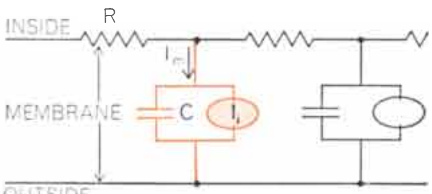
In 1943 Warren S. McCulloch and W. S. Pitts of the Massachusetts Institute of Technology developed an abstract and highly simplified model of the basic element of a biological nervous system: the neuron, or nerve cell. It was in effect a finite automaton with only two possible states—firing or not firing. By combining these modules, or formal neurons, they set up models of nervous systems, and later S. C. Kleene of the University of Wisconsin proved a general theorem characterizing the kinds of behavior that can be expected of networks of McCulloch-Pitts neurons.

A MATHEMATICAL MODEL OF THE NERVE IMPULSE

The most frequent subjects of mathematical modeling in the biological sciences are neurons, or nerve cells, and networks of neurons. Some models, as Edward F. Moore indicates in the accompanying article, leave biological fact far behind. Other models, however, seek to explain the actual functioning of nerve cells or fibers. The classic example is the model of the nerve impulse that was developed by A. L. Hodgkin and A. F. Huxley of the University of Cambridge.

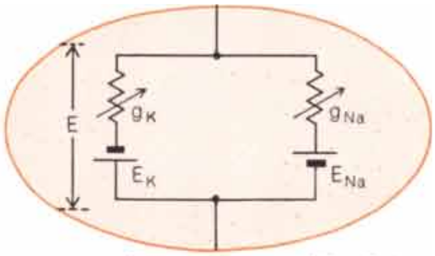


NERVE IMPULSE is a wavelike variation of potential difference across the nerve-axon membrane. Sodium ions enter the axon during the rising phase of the impulse and potassium ions leave during the falling phase.



$$I_m = \frac{1}{R}\frac{\delta^2 E}{\delta x^2} = C\frac{\delta E}{\delta t} + I_i$$

CABLE ANALOGUE of nerve axon has longitudinal resistances, membrane capacitances and ionic-current elements (I_i).



$$I_i = \bar{g}_{Na} m^3 h(E - E_{Na}) + \bar{g}_K n^4 (E - E_K)$$

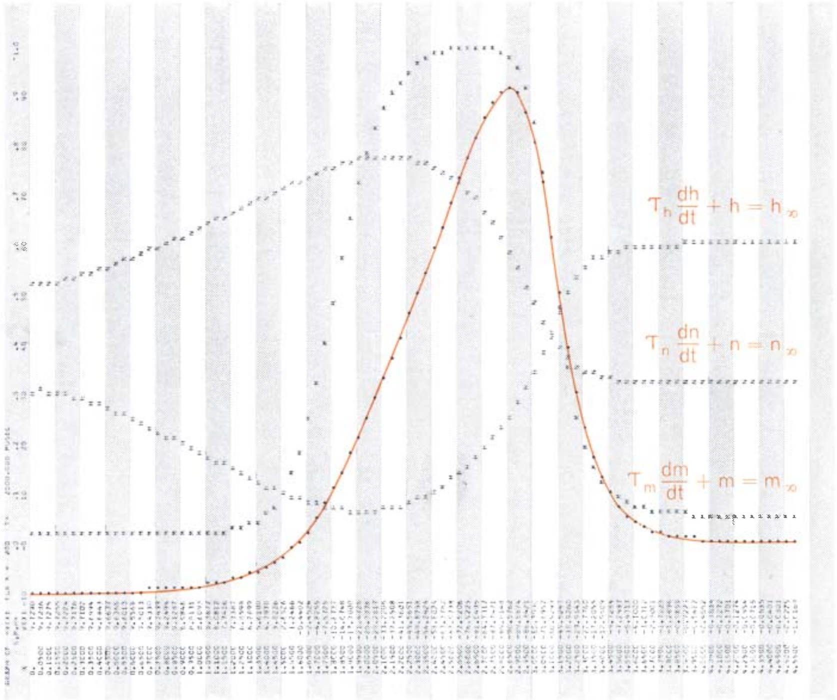
IONIC-CURRENT ELEMENTS have potassium and sodium batteries, each with a variable resistance. E is membrane potential.

The signal that travels down the cable-like axon of a nerve cell is an electrical pulse about 100 millivolts in amplitude and one millisecond in duration. The impulse is propagated at constant velocity and shape by a mechanism that regenerates the pulse at each point along the axon; the energy for this regeneration comes from the distribution of ions between the inside and the outside of the axon. Potassium ions are more concentrated inside the axon, and if the axon membrane were permeable only to potassium, the potential difference across it would be the potassium equilibrium potential (E_K) of about 75 millivolts, with the inside of the axon negative. Sodium ions are more concentrated outside, and if the membrane were permeable only to sodium, the voltage across it would be the sodium equilibrium potential (E_{Na}) of about 50 millivolts, with the inside of the axon positive. What happens is that the permeability of the membrane to ions undergoes dramatic changes in response to changes in potential. In its resting state the membrane is moderately permeable to potassium and the potential is about -60 millivolts. If an artificial stimulus or an advancing nerve impulse reduces this potential, the membrane momentarily becomes highly permeable to sodium ions, which enter the axon; the potential in this active region momentarily becomes positive. Almost immediately, then, the sodium permeability is shut off and the potassium permeability increases, restoring the membrane potential to its resting value as the nerve impulse moves on down the axon.

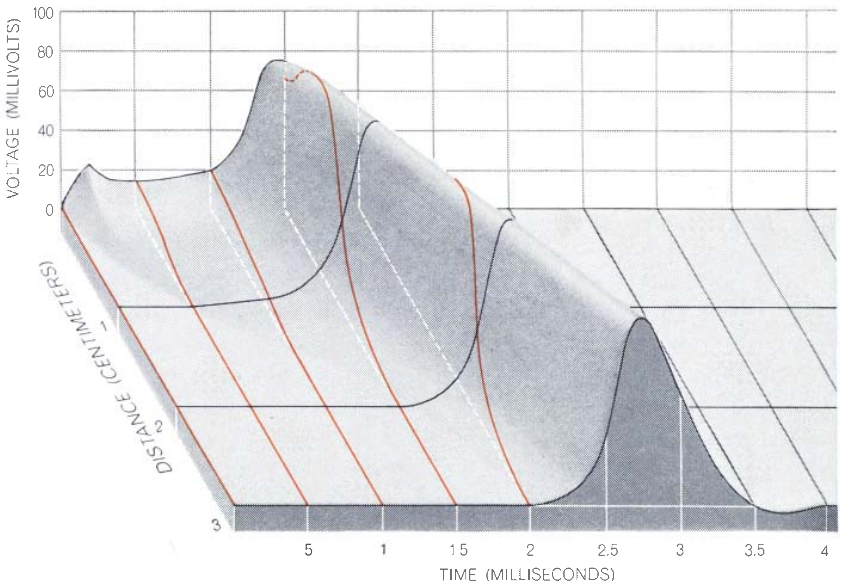
A number of investigators in Britain and the U.S. established that the propagation of the nerve impulse, or action potential, is essentially an electrical process described by the nonlinear partial differential equation shown in the middle illustration at left. The equation says that the current (I_m) across any segment of the membrane (*color*) equals the difference between the longitudinal currents flowing into and out of that segment, and that this in turn is equal to the membrane capacitance current plus the current carried by ions moving through the membrane. The dependence of this ionic current on the membrane potential was still unknown until 1949, when Hodgkin and Bernhard Katz discovered the effect of the sodium-potassium sequence on the rising and falling phases of the impulse. In 1952 Hodgkin and Huxley, using a technique originated by K. S. Cole at the Marine Biological Laboratory at Woods Hole, Mass., measured the ionic current when the membrane potential was held constant. From these data they deduced the laws governing the dependence of sodium and potassium permeability on membrane potential, and they formulated the laws in a set of differential equations.

The active nonlinear element that accounts for the ionic current is shown in the bottom illustration at left. The equation with it shows how the ionic current is the product of the electrochemical driving force for each ion and its specific conductance. The sodium and potassium conductances depend on three dimensionless, experimentally determined variables: m, h and n. These variables obey the nonlinear differential equations shown in the top illustration at right. The time constants (τ) and steady-state values in the equations are functions only of the membrane potential. Five equations—these three and the two previously described—together form the Hodgkin-Huxley model.

If one assumes that a potential pulse travels with a constant velocity, solution of the equations yields the wave form of the impulse and produces an adequate description of other observed phenomena. In order to inspect the transient buildup of the pulse from a stimulus, however, one needs a more general solution. Recently Fred Dodge, James Cooley and Hirsh Cohen at the International Business Machines Research Center have carried out a computer calculation in which they replaced the five equations with difference equations. One of their results is the set of curves in the top illustration, showing the relation between sodium and potassium conductances and potential. The three-dimensional graph in the bottom illustration exhibits the formation of the pulse as it tends toward a fixed shape and constant velocity. This new method of solving the equations should also make it possible to study the propagation of the nerve impulse in axons that branch or otherwise change in geometry.



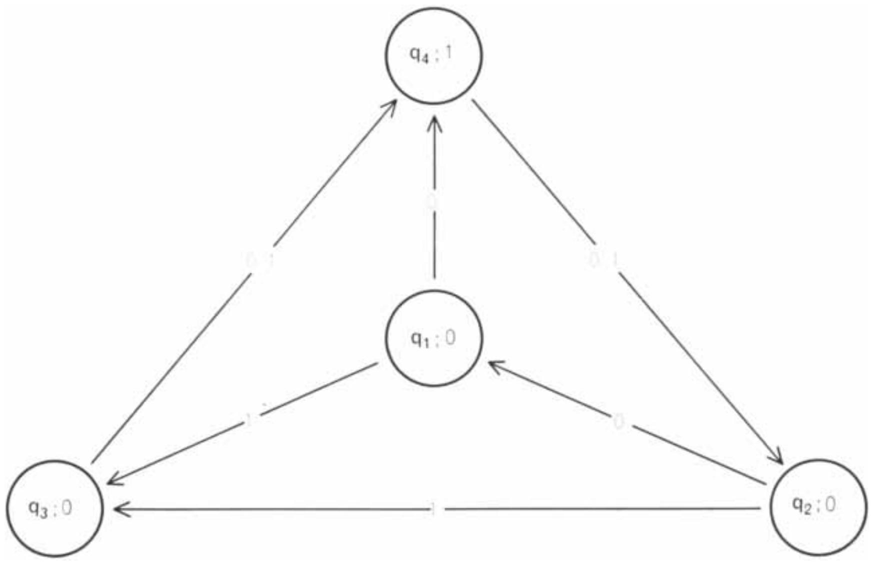
COMPUTER PRINT-OUT shows how the membrane potential and effective conductances of sodium and potassium vary along the axon at a fixed time. The distance scale runs from 0 (*left*) to 4.55 centimeters. The potential curve is delineated by asterisks (with a curve added for clarity) on a scale of 0 to 90 millivolts, 0 corresponding to the membrane's resting potential of —60 millivolts. The three curves consisting of letters are the variables representing changes in ionic conductance of the membrane: *M* (sodium activation), *H* (sodium inactivation) and *N* (potassium activation). Their values lie between 0 and 1. These variables obey the nonlinear differential equations shown with each curve. The voltage-dependent time constants and the steady-state values in each equation were obtained experimentally.



THREE-DIMENSIONAL GRAPH of a nerve impulse moving out in time and space was constructed from a series of curves generated by a computer, which gave the voltage as a function of distance along the axon at successive time intervals (*colored curves*). The black curves show the voltage as a function of time. The graph shows how a short stimulus (.2 millisecond) triggers a change in potential that hovers around the threshold and then flares into an action potential, and how the nerve impulse attains its constant shape and velocity.

PREVIOUS STATE	PRESENT STATE	
	0	1
q_1	q_4	q_3
q_2	q_1	q_3
q_3	q_4	q_4
q_4	q_2	q_2

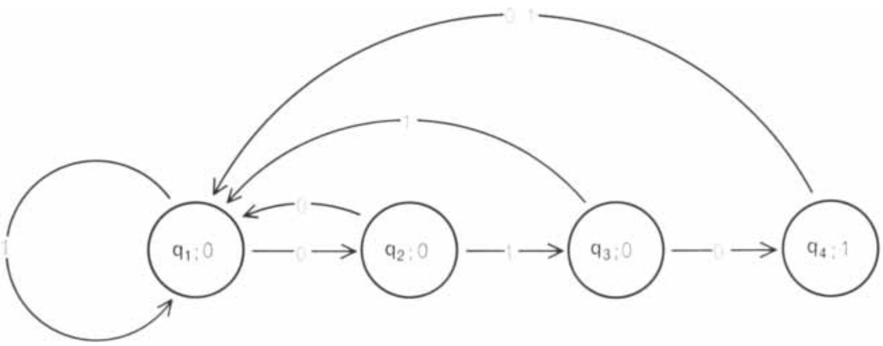
PRESENT STATE	PRESENT OUTPUT
q_1	0
q_2	0
q_3	0
q_4	1



FINITE AUTOMATON is an idealized machine or part of a machine; in automaton theory it is the basic building block of systems ranging from neural networks and computer circuits to self-reproducing machines. An automaton can be described by two tables. One (*top left*) shows the change in state in a unit of time depending on input (*color*). The other (*top right*) shows the output associated with each state. The description can be diagrammed (*bottom*). Vertices give the states and outputs; edges give transitions for each input (*color*).

PREVIOUS STATE	PRESENT STATE	
	0	1
q_1	q_2	q_1
q_2	q_1	q_3
q_3	q_4	q_1
q_4	q_1	q_1

PRESENT STATE	PRESENT OUTPUT
q_1	0
q_2	0
q_3	0
q_4	1



COMBINATION LOCK can be described in finite-automaton terms. The input sequence is the combination. In this simple example the combination is 0, 1, 0; output 1 is "unlocked."

Kleene's theorem has since been applied directly to arbitrary finite-state machines, without considering neurons. What began as an attempt to analyze the nervous system therefore led eventually to theoretical advances in logic and electrical engineering.

The British logician A. M. Turing took a different approach to the "thinking machine" problem. Without making any assumptions about the physiology of the brain, he sought to define in logical terms an automaton that could in principle carry out any precisely definable computation. A Turing machine works by performing a very large number of exceedingly simple operations. It is an automaton with a finite number of states but supplied with an infinitely long "tape." Its instructions are written on a finite portion of the tape; given enough time and enough steps, the "head" of the machine reads these instructions and prints out its answer on more of the same tape. Turing showed that a "universal" machine could even be designed to perform any possible computation. If it is supplied with descriptions of a task and of a machine that could perform that task, it will figure out how to do the job itself and proceed to do it.

Both the McCulloch-Pitts neurological model and the more abstract Turing machine have stimulated interesting speculations on the nature of thinking and on the extreme capabilities of machines. Automaton theorists have, for example, tried to mimic the ability of biological systems to repair themselves and to perform reliably in spite of having unreliable components. Von Neumann showed how one could make a machine that would function properly, even if some of its parts failed, by introducing redundancy: by providing multiple logic elements in a computer, for instance, and multiple pathways among them. Both self-repair and redundancy are of obvious importance in computer and other electronic technologies and are currently under serious investigation.

The fact remains that a biological neuron is not a simple on-off switch that is either firing or not firing at any given time, and that computers are not "thinking machines." There are probably better ways to build computers than to imitate what we think we understand of the brain, and better ways to improve that understanding than by oversimplifying the behavior of neurons. It has sometimes seemed to me that the effort to design oversimplified neurons

A MODEL OF ENERGY TRANSFER IN PREDATION

Ecology, the study of interactions between organisms and their environment, is a field of biology that has seen an increasing application of what might be called "constructive" mathematics, or mathematics designed to create models that in turn lead to predictions and biological insights. This is a brief account of one such piece of work, an investigation of predation by L. B. Slobodkin of the University of Michigan. Slobodkin experimented with laboratory populations of the minute freshwater crustacean *Daphnia*, measuring the caloric content of a population, of its food (algae) and of the *Daphnia* taken as "prey" (simply by removing them from the tank) at various rates. He reasoned as follows:

If the energy income, or input, per unit time to a steady-state population is I, and if in the absence of predation the "standing crop," or caloric content of the population, is P, and if the cost in calories to maintain one calorie of crop for one day is c, then

$$I = cP\,.$$

If there is predation, the standing crop P' will be reduced; assuming that the energy input is constant, there will be a change (Δc) in the cost, so that under predation

$$I = (c + \Delta c)P' = cP' + \Delta cP'\,.$$

The last term in the equation represents the effect of predation. If one considers this effect in a different way, then

$$I = cP' + \sum \frac{Y_i}{E_{pi}}\,.$$

The new expression is a summation of as many different kinds of yield (Y_i) as are involved, divided by E_{pi}. The latter is a dimensionless number, a "regression coefficient" inserted to avoid converting I into Y without taking into account the events implied by the switch from algae (I) to *Daphnia* (Y). Inspection of the equation shows that a high E_{pi} will make for a low drain on the population for a given amount of yield. Having measured I, P' and Y_i, Slobodkin solved equations for c and the various values of E_{pi}. High E_{pi} values turned out to be associated with older *Daphnia*. This implied that E_{pi} should be equivalent to some biologically meaningful expression.

Now, if one assigns to each dying animal the energy required to replace it, the sum of the replacement costs of all animals dying in a given interval in a steady-state population should precisely equal the energy income to the population in that interval. So if K_i is the cost of replacing an animal of age i, and if D_i such animals die in a unit of time, then in the absence of predation

$$I = \sum K_i D_i = cP\,,$$

and in the presence of predation

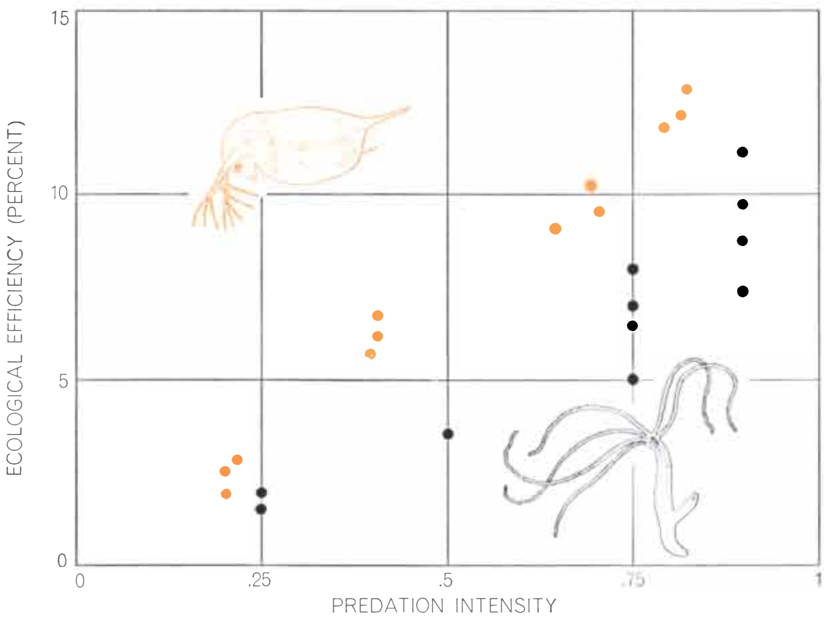
$$I = \sum K_i' D_i' = cP' + \Delta cP'\,.$$

Now suppose only one kind of prey is taken, k. Then by manipulating equations one can write

$$E_{pk} = \frac{Y_k}{\sum K_i' D_i' - \frac{P'}{P} \sum K_i D_i}\,.$$

This equation implies that predation on animals that are about to die anyway minimizes the difference between the two death distributions and is therefore best for both the predator and the prey. It provides the predator with a maximum steady yield and at the same time enables the population to withstand predation. In nature, then, both predator and prey might be expected to evolve toward a relation in which the animals taken as yield will have maximum E_{pi}. For the predator this means taking the biggest and weakest animals it can get; for the prey it suggests a life pattern in which the causes of death other than predation accumulate at the age level normally taken by the chief predator.

The regression coefficient E_{pi}, which Slobodkin calls "population efficiency," leads to new questions about actual predator behavior and the mechanism of aging; it suggests a theory even though it makes no unique numerical prediction. In this it is unlike another concept of efficiency, Y/I, or "ecological efficiency," which has a fairly constant maximum value in various populations [*see illustration*]. There is as yet no explanation for this value, although it is predictable. The ratio Y/I, in other words, still awaits constructive mathematical analysis.



ECOLOGICAL EFFICIENCY tends to reach a maximum value of 7 to 13 percent, whether in *Daphnia pulex* (color) or in species of the coelenterate *Hydra* (black).

as a step toward "thinking machines" is comparable to an early attempt to build artificial feathers to enable men to fly: a bird's flight reveals some fundamental principles of aerodynamics that are pertinent to aircraft design, but the details are significantly different. As for describing brain function in logical terms, that appears to be far beyond our present capabilities. Von Neumann suggested that the simplest possible description of a specific mode of operation of the brain may be a diagram of all the nerve connections themselves! There may be a mathematical logic that can explain the brain, but it is surely orders of magnitude more complex than anything mathematicians have yet devised.

Is there perhaps some characteristic of living things that is more amenable to logical analysis? Self-reproduction is a good possibility. Certainly it is the most primitive aspect of life. Most organisms do not "think," and many have no nervous system at all, but all organisms do reproduce themselves. There is a good chance, then, that self-reproduction will turn out to be much simpler logically than thinking is. If one could uncover a logical scheme of reproduction, one should at least be able to identify the important problems in the process and perhaps even some of the ways in which those problems can be solved; this might in turn tell the biologist what to look for and perhaps throw light on some of the biological processes that are now under intensive investigation.

Von Neumann was the first to consider in detail how to make a machine reproduce itself. The task he set himself, as he once stated it, was: "Can one build an aggregate out of [simple] elements in such a manner that if it is put into a reservoir, in which there float all these elements in large numbers, it will then begin to construct other aggregates, each of which will at the end turn out to be another automaton exactly like the original one?" He proceeded to show that this was feasible: that a properly instructed machine placed in a "stockroom" environment would wander around assembling the parts with which to reproduce itself, and that there would in time be two and then four, eight and 16 machines and so on as long as the parts held out and space was available.

At first this sounds quite impossible and even ridiculous. Is it not, however, perhaps just a matter of degree? If a small crystalline "seed" composed of a few molecules of a substance is introduced into an environment of many of the same molecules at the appropriate temperature and pressure, the seed causes more molecules to be assembled in the same crystalline configuration. Seen in this light, crystal growth is self-reproduction. So is the closing of a zipper: two little hooks are brought together in an environment consisting of more hooks, arranged in a row, and a runner; all the other hooks join in sequence to form a series of two-hook "machines," a sort of one-dimensional crystal.

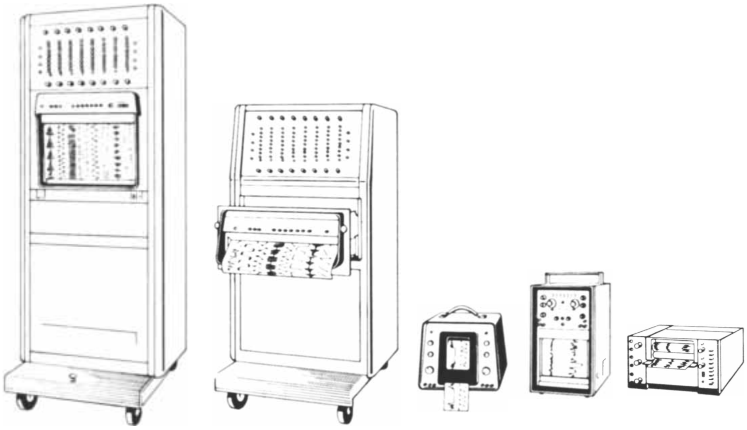
These are, to be sure, trivial examples, because each involves one simple change in state: from amorphous to crystalline, from open to closed. How about the punched card of a business machine? It is a "machine" and it reproduces itself—with the help, to be sure, of a complex environment: the card-punching equipment, which supplies most of the organization required for duplicating the card. (Note, however, that whereas some one-celled organisms can reproduce in a simple nutrient medium, higher organisms may be dependent on a complex environment that contains, for instance, such complex elements as vitamins.) The real problem, then, is to produce a complicated machine in a simple environment—a machine with a large number of parts, perhaps, but not very many kinds (or "states") of parts. This is what von Neumann accomplished with his model of self-reproduction.

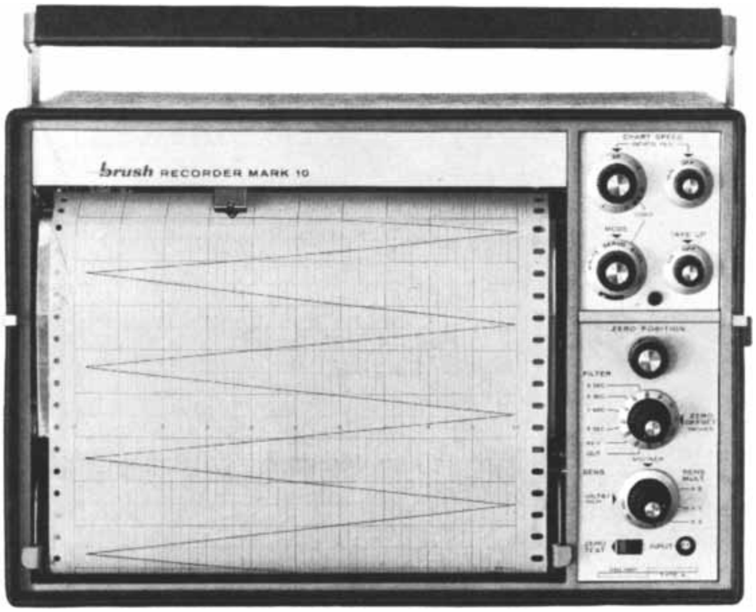
One might still ask if von Neumann's model bears any significant relation to biological reproduction. One of the first logical difficulties he encountered provides a good example of such a relation, as does the way in which he solved it. He quickly realized that the instructions that tell the machine how to construct a copy of itself cannot be complete. To be complete they would have to describe not only the automaton but also the instructions themselves; there would be a blueprint of the blueprint, and this could go on in an infinite regress. The way to get around this problem is to provide two machines that operate on the blueprint in two ways. One is a blueprint-copier (*C*). The other is a blueprint-obeyer (*O*). They are combined, along with a sequencing device (*s*) that turns each on at the proper time, with the blueprint describing all three elements (B_{C+O+s}). The complete machine can be symbolized as $C+O+s+B_{C+O+s}$. Presented with the blueprint for the total machine, *C* copies it and *O* follows it to build *C*, *O* and *s*.

Recent findings in the chemistry of genetics reveal some striking parallels between von Neumann's elements and processes in the living cell. *B* is a set of genes composed of deoxyribonucleic acid (DNA), which encodes hereditary characteristics. *C* is the enzyme DNA polymerase, which catalyzes the replication of a strand of DNA and so

X	Y	Z	O	X	Y	Z	O
X	Z	Y	X	X	Z	Y	X
O	O	O	O	O	O	O	O
X	Y	Z	X	Y	Z	O	O
X	Z	Y	X	Z	Y	X	O

TESSELLATION is a plane subdivided into square cells. This array is a configuration of 40 cells, shown in states designated *X*, *Y*, *Z* and *O* (quiescent). The total configuration contains three "copies" of a seven-cell configuration (*color*). A copy must be a "disjoint subset"; the fourth set like the others (*black outline*) is not a copy because it is not disjoint.





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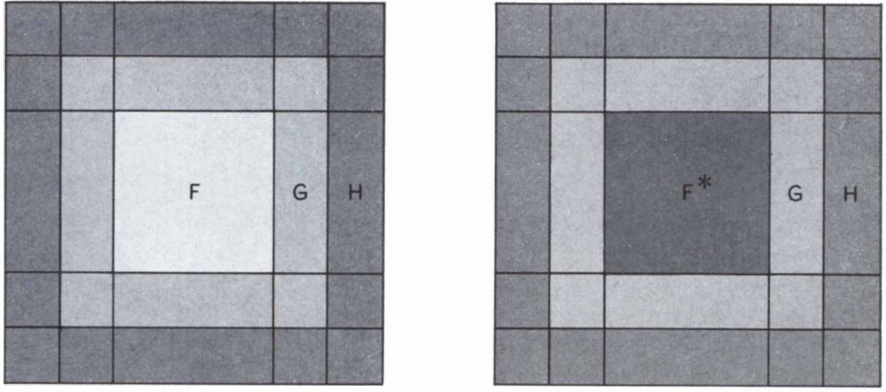
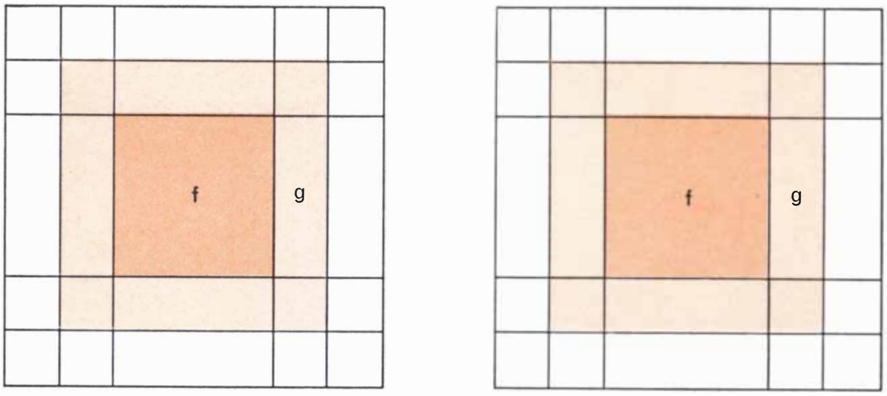


CONFIGURATIONS of two arrays of the same size are shown as of time T. The configurations of the two inner arrays are different from each other. (The drawings show inner arrays of one particular size but they could be any size.) The configurations marked G, the hollow squares of neighbors of the inner arrays, are similar to each other. So are the H's.



NEW CONFIGURATIONS of the arrays shown at the top of the page are shown here at $T + 1$. The inner arrays have been mapped into a new configuration that is the same for each of them. The intermediate arrays are also similar to each other. The state of the outer arrays cannot be specified. Two arrays that differ as in the illustration at the top of the page and are followed by identical arrays, as here, are said to be "mutually erasable."

copies the genes. O is the system of messenger ribonucleic acid (RNA), enzymes and ribosomes that assembles amino acids according to the prescription of the DNA to synthesize enzymes and other proteins and thereby build the new cell.

Von Neumann's early models were "kinematic" in that they dealt with physical parts in motion, but they were not actually built. Some less general kinematic models have been demonstrated, however. In one, designed by Homer Jacobson of Brooklyn College, the self-reproducing machine is a train of toy railroad cars; the cars are the individual parts and control is accomplished by circuits mounted in each car. When a train of different kinds of cars arranged in a particular sequence is placed on a track layout, it moves around bumping into unattached cars, rearranging them when necessary by using sidings and coupling them in its own image. Jacobson designed one such train model that would work for arbitrary trains of any length, and he actually built a simpler demonstration model in which a train two cars long can reproduce itself.

The British geneticist L. S. Penrose took a different approach, stimulated in part by his speculations on the manner in which the genetic material might replicate. In its most elementary form Penrose's machine has two kinds of part, A and B, simple cutouts shaped so that they can hook together in either of two ways. If an AB machine is placed on a tray with a number of A's and B's and the tray is shaken, a number of new AB units are formed; the first machine acts as a seed [*see illustration on page 148*]. If the seed is a BA machine, the reproduced pairs will be more BA's.

Kinematic models have the virtue of realism and dramatic impact, but they are exceedingly hard to deal with mathematically. In his later work von Neumann turned to an abstract model; he discarded the hardware, thereby avoiding all problems of mechanical movement, fit and manipulation and setting himself a more purely logical and mathematical task rather than one in mechanical or electrical engineering. For his stockroom he substituted a mathematical environment: a tessellation, or plane subdivided into square cells. In each cell he placed one of his elementary parts: a finite-state machine. Von Neumann's machines have no inputs or outputs but only a number of permissible states. The list of these states and the rules governing transitions from one state to another are the same for all the cells, although different cells can be in different states at any one time. Each machine is deterministic and synchronous: at each integer-valued time T (except in the initial condition, or at $T = 0$) the state of each cell depends only on its own and its neighbors' states at time $T - 1$. There is a special state known as the "quiescent" state; all but a finite number of cells are in this state, which has the special property that if any cell and all its neighbors are quiescent at time $T - 1$, that cell will be quiescent at T. The entire system—tessellation space, cell machines, allowable states and transition rules—is called a "tessellation structure." A finite block of cells, or "array," is called a "configuration" when the states of its cells are specified.

What have all these definitions to do with machines and reproduction? Consider the entire tessellation space as an extremely abstract environment—one in which space and time are quantized and both motion and other forms of gradual change are replaced by successions of discrete states, the transitions between which are prescribed by definite rules. Within this environment there are configurations of square cells; these are our "machines," and it is these machines that can be made to be self-reproducing. The individual cells are the elementary parts—perhaps molecules. Their changing states can be thought of as different quantum states—energy levels or chemical-activation states—and geometrical positions. The rules for transitions between states are the physical and chemical laws of this environment that determine how the cells change and relate to one another. The quiescent-state cells are unused raw materials, and the rule for quiescent states says in effect that no cell separated from a configuration will suddenly flare into activity; a machine "reaches out" to surrounding materials through local action only.

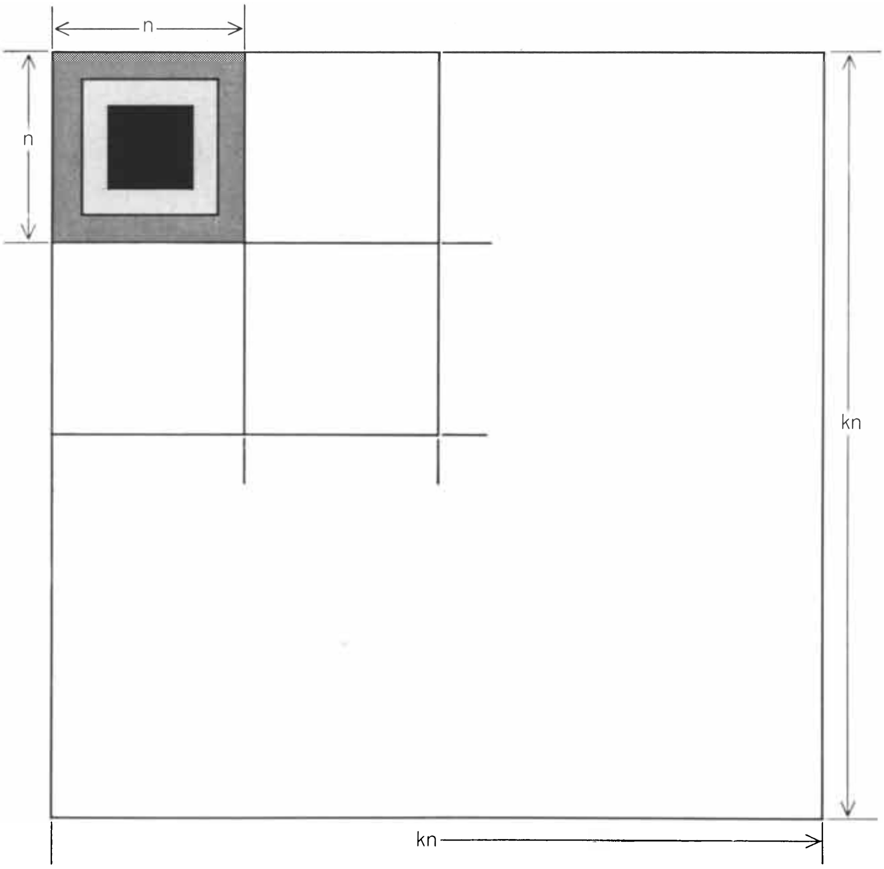
The problem, therefore, is to build a tessellation structure with individual cells that have a small number of states (fairly simple parts, in other words), to choose the transition rules and then to arrange cells into a configuration that will in time reproduce itself. It is a task somewhat like writing a program for a digital computer. Von Neumann set a further requirement: Each configuration must contain a universal Turing machine. He then worked out most of the details for a self-reproducing configuration of some 200,000 cells with 29 permissible states. Since his death in 1957 others have continued to work with tessellation models, examining the details of their construction and attempting to draw conclusions, in the form of provable theorems, as to the logic of reproduction.

One question that I asked was: How fast can a population of self-reproducing configurations grow? The answer is that it cannot grow exponentially—cannot keep doubling with each generation, for instance. The population of a two-dimensional tessellation cannot at any time be larger than the square of the time. Let us formulate that statement as a theorem: If a self-reproducing configuration is capable of reproducing $f(T)$ offspring by time T, then there exists a positive constant k such that $f(T) \leq kT^2$. (The symbol $\leq$ means "is less than or equal to.")

The proof is as follows: Let c be the self-reproducing configuration and let the smallest square array able to contain a copy of c be of size $D \times D$. Then at each time T the total number of nonquiescent cells is at most $(2T + D)^2$, since the square array can only grow by one cell on each side with each unit of time. If r is the number of cells in c, then

$$f(T) \leq \frac{(2T + D)^2}{r}.$$

This inequality expression can be simplified in a few more steps to reach



"GARDEN OF EDEN" THEOREM (*see text*) is proved with the help of this diagram. The small array of size $n \times n$ has an erasable configuration. The large array is of size $kn \times kn$. (The integer k is 4 in the illustration but would actually be much larger.) This drawing shows the array as of time T. At $T + 1$ it will be reduced in size to $(kn - 2) \times (kn - 2)$.

$$B = \frac{M}{R^3 \cdot \cos \lambda^2 \cdot \sqrt{1+3\sin^2\lambda}}$$

```
PROCEDURE
CALCULATE FIELD STRENGTH;
BEGIN
REAL B; COMMENT B := MAGNETIC FIELD STRENGTH;
B := M/(R↑3*CØS (LAMBDA↑2)*SQRT (1 + 3*SIN (LAMBDA)↑2))
END OF CALC. FIELD STRENGTH;
```

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- Radiation dosage per unit area received by an object traveling through the geomagnetic radiation belts.
- Dynamical behavior of the Apollo vehicle in simulated flight, including the effects of errors.
- Statistical analysis of the probability of locating suitable lunar landing sites as a function of the types of surfaces on the moon.

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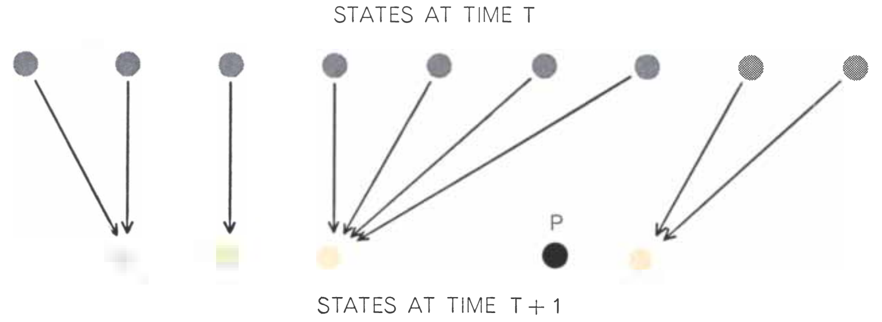

NUMBER OF POSSIBLE STATES for the array illustrated on page 158 decreases between time T and $T+1$ as a result of erasure and of shrinkage at the boundary layer (*see text*). It can be shown that the loss by erasure is larger than the loss by shrinkage. There will therefore be an extra state at $T+1$; it is P, the Garden of Eden configuration.

the conclusion of the theorem. The situation is Malthusian: the population is limited by the space available, since the nonquiescent region has a velocity of propagation that cannot exceed a fixed constant.

Another important question is this: Can every configuration reproduce itself? It turns out that there are configurations that cannot occur except at the outset, at $T = 0$. Such a configuration is nonconstructible, in the sense that there is no configuration that can give rise to it by means of the specified rules of transition. John W. Tukey of Princeton University suggested calling such a block of cells, which can have had no predecessor state, a "Garden of Eden" configuration. Since such configurations cannot be produced by any other configuration, they cannot reproduce themselves. An investigation of the conditions under which they can occur therefore helps to define fundamental limitations on the ability of machines to reproduce themselves.

These conditions involve the ability to perform something called "erasing." When a blackboard has been erased, there is no way of telling what was written there. Computer designers apply the term "erasing" to an operation on computer memory elements that puts them into a standard state regardless of what was stored there before the erasing. In general, erasing is an irreversible process; after it occurs it is impossible to determine the preceding states from which the present state could have arisen. There must, then, have been at least two possible preceding states that could go through transitions putting them into the same present state. In the case of tessellation structures it is necessary to define erasing somewhat more formally to make sure that the information as to past state is really erased and not merely shifted away to neighboring states.

To establish such a definition consider the two configurations in the upper illustration on page 156. The configurations of the two nine-cell inner arrays are different at a given time; call them F and F^*. At the same time the configurations of the intermediate arrays—the hollow squares of immediate neighbors of the inner arrays—are both G; that is, one intermediate array is a copy of the other. The same is true of the configurations of the outer arrays: both are H. Now consider the configurations into which these arrays will have been mapped after one unit of time [*see lower illustration on page 156*]. If, in these $T+1$ arrays, the configurations f and g of both the inner and the intermediate arrays are copies of each other, then the two original configurations are said to have been "mutually erasable"—they were different and have become alike. Note that one cannot specify the final configuration of the outer array because the states of its cells are affected by exterior cells that have not been considered in this definition; its function has been simply to "protect" the intermediate array. A configuration, then, will be said to be erasable if there is another configuration such that the two are mutually erasable. It further follows from this definition that if an erasable configuration of any shape is contained within a rectangular array, that rectangular array is also erasable. As a result one can consider this problem in terms of square arrays, which are easier to work with than those of irregular shape.

It is now possible to state a theorem: In a tessellation structure for which there exist erasable configurations, there

exist Garden of Eden configurations. (Although one might construct a tessellation without erasable configurations, such a structure would be essentially trivial.)

I shall sketch the intuitive proof of this theorem rather than give the detailed proof. Let n be an integer such that there is an array of size $n \times n$ that has an erasable configuration. Then consider a larger array of size $kn \times kn$ [*see illustration on page 158*]. Each of the k^2 arrays of size $n \times n$ is the proper size to contain a copy of an erasable configuration; k is chosen to be large enough so that there will often be many such erasable configurations in the large array. If A states are permissible for each cell, then the whole array can have $A^{(kn)^2}$ possible configurations at time T. Consider next the array into which this first array is mapped after one unit of time. Remember that one cannot specify the states of the outer hollow square at $T+1$. There are therefore fewer possible states available to the $T+1$ array: $A^{(kn-2)^2}$.

Now, if in the original $kn \times kn$ array there was one $n \times n$ array with an erasable configuration, then two possible states—those of the erasable configuration and of the one mutually erasable with it—will both map into only one possible state at $T+1$. If there were two copies of an erasable configuration, their four possible states will map into only one possible state. In general, if there were s copies of the erasable configuration at time T, there will be 2^s states mapping into one state at $T+1$. This situation is diagrammed in the illustration on page 160. Now it is only necessary to show that the loss in the number of states due to erasure must be larger than the loss due to shrinkage of the outer boundary cells—that is, the loss due to the difference between $A^{(kn)^2}$ and $A^{(kn-2)^2}$.

Consider the logarithms of the numbers of states rather than the numbers themselves. The logarithm of the ratio that indicates boundary-layer loss increases approximately linearly with k. But the logarithm of the number of states lost by erasure increases with the number of erasable configurations, and therefore approximately with the area of the array, or with the square of k. For a large k, then, more states will surely be lost by erasure than by shrinkage at the boundary layer. Therefore there must exist a state P at time $T+1$ that cannot be reached from any of the states at time T.

This state P is the Garden of Eden

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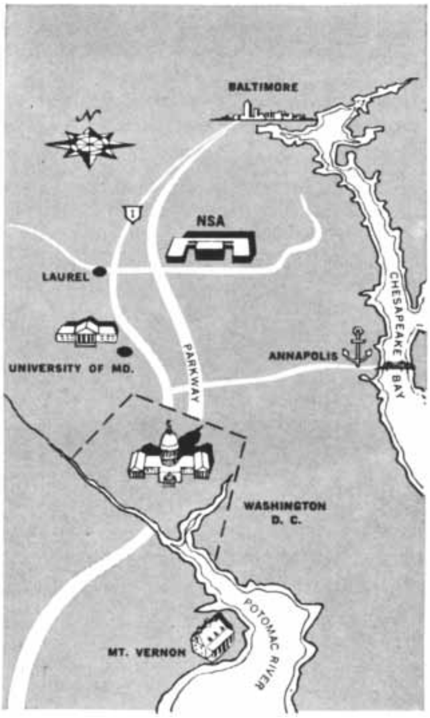

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configuration of the theorem. It is a conceivable state but one that cannot be reached from any previous state. It corresponds to a machine that can be described as an arrangement of the available parts but that cannot be built out of those parts. Since it is a machine that cannot arise by reproduction, it must be a machine that cannot reproduce itself.

The two theorems I have discussed may suffice to give some idea of the manner in which a process such as reproduction can be abstracted and made amenable to mathematical treatment. This is not to suggest that there is necessarily a close relation between a tessellation model of self-reproduction and biological reproduction; that certainly remains to be shown. Yet, as I proposed earlier in this article, it seems likely that manipulation of machine models should at least help by identifying difficulties or establishing criteria for biological processes.

To take just one example: Life on earth is now generally assumed to have originated through the chance interaction of nonliving materials. How likely would this be? Perhaps one could learn from tessellation models just how complicated an assemblage of parts must be in order to have the property of self-reproduction and the further property of being able to undergo evolution into more complicated descendants. Jacobson, the investigator who worked with toy-train machines, made a start in this direction, characterizing complexity in terms of "bits" of information.

Even if machine self-reproduction finally proves quite inapplicable to biology, it may nevertheless be of surpassing interest—for its own sake. Let us return for a moment to von Neumann's stockroom model of a self-reproducing machine. What if we devise such a machine not for an arbitrary stockroom but for some natural environment? Such a machine—really an artificial living plant—would make its own parts out of natural materials and then assemble those parts to reproduce itself. In the process it would extract substances from its environment and refine and concentrate them. A plant designed to reproduce in the oceans, for instance, might build itself largely out of magnesium, which is present in sea water, and could be harvested for its magnesium content. Nobody has yet done the engineering design work required to build such a machine, but I think it will someday be built.

Mathematics in the Social Sciences

Certain social sciences such as economics deal with facts that are usually represented by numbers. New techniques for relating these numbers point toward the mathematical analysis of entire societies

by Richard Stone

Some 75 years ago the American economist Irving Fisher stated that the entire world's literature contained scarcely 50 worthwhile books and articles on mathematical economics. Today the situation is different, not only in economics but also in all the other social sciences: each year sees thousands of additions to a mountain of mathematical literature. The reason is a growing appreciation of the advantages that come from expressing in mathematical terms concepts that were once dealt with only verbally.

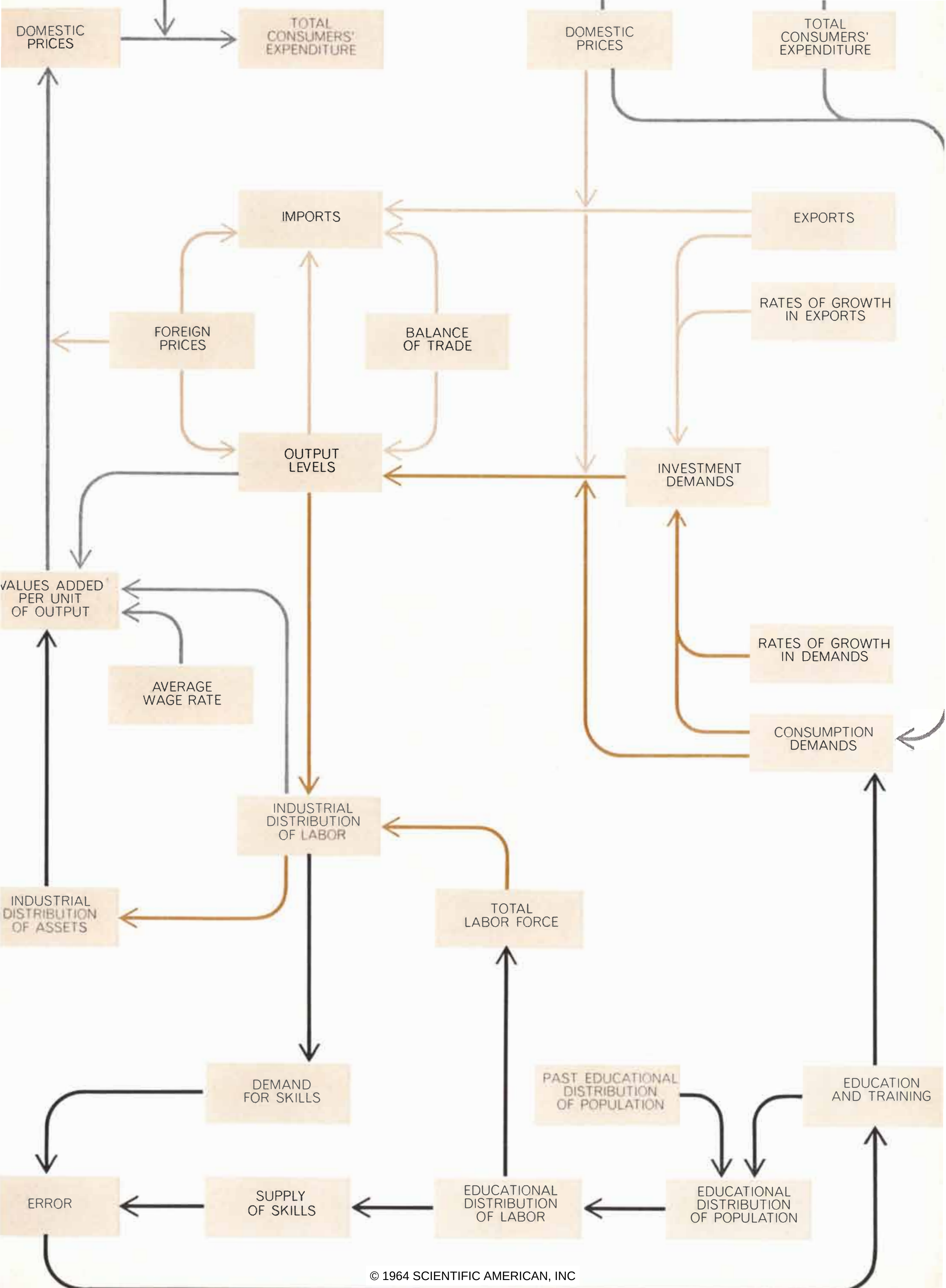
The progress has been greater in some areas of study than in others. The entire subject matter of demography and economics, for example, is so aggressively quantitative that a wider application of mathematics to these fields has been inescapable. But in every one of the social sciences it has become increasingly evident that an exclusively verbal description of complex systems and their interrelations—and more significantly the framing of theories about such systems—results in generalizations that are difficult to analyze, compare and apply. These difficulties are greatly reduced when mathematical expressions are substituted for words. For one thing, a number of problems that had seemed to be completely unrelated—for instance the analysis of educational systems and the programming of capital investments—prove to be mathematically identical. For another, even in subjects whose concepts are rather vague and in which precise information is hard to find, mathematics can provide a means of obtaining valuable insights.

At the risk of enormous oversimplification it can be said that the social sciences, for all their diversity, are concerned with only two major areas of investigation. The first of these is the precise description of how social systems work and how their different parts are interrelated, whether the subject is cross-cousin marriage in a tribal society or the contribution of the steel industry to the total output of a technologically advanced nation. Studies of this kind investigate and analyze structure. The second area of investigation is concerned with control, that is, the effect of conscious aims on the operations of social structures and an examination of the rational processes that underlie the formulation of policy. These studies investigate and analyze decisions.

In both fields the same kinds of mathematics are applied to similar problems. Where empirical analysis of discrete observations is necessary much use is made of finite mathematics: in particular matrices, matrix algebra and difference equations. Where, on the other hand, purely theoretical analysis is desired and the data provide a continuum rather than discrete "bits," the use of the infinitesimal calculus—in particular differential equations—has many advantages. The examples that follow, drawn from the investigation of many kinds of structure, demonstrate the application of both kinds of mathematical method.

Among its many investigations the discipline of demography requires analysis of the structure and probable development of populations. To project the future structure of any population, assuming a constant pattern of births and survivals, we need three sets of data: (1) the numbers of people of different ages alive at a certain date; (2) the numbers surviving over a chosen time interval immediately following that date, and (3) the numbers born during that same time interval to the members of the different age-groups. This third set of data, by the way, exemplifies the kind of imprecision that can creep into quantifications of social science. In real life husbands and wives often belong to different age-groups, and the wife's childbearing capacity is a function of this factor as well as of her own fertility. In

FOUR MAJOR CIRCUITS, constituting a mathematical model of the British economy, are made to interact in the manner illustrated on the opposite page. At the heart of the model lies the circuit of real flows (*solid colored arrows*). Consumption demands and the rate of growth in these demands, together with exports and their rate of growth, combine (*middle right*) to determine the level of investment demands necessary to produce the required growth of output. Output levels themselves are composed of three "final" demands (for consumption, investment and exports) plus all intermediate demands for raw materials and fuels. Given the levels of output and the total labor force, industrial distributions of labor and capital are determined from the consideration that resources be used efficiently. The foreign-trade circuit (*light colored arrows*) has a similar two-part start and, in addition to the obvious interactions of foreign prices and balances of trade on imports, has its effects both on domestic output levels and on domestic prices. The price circuit (*gray arrows*) shows an immediate interaction with consumption demands and a feedback, derived from considerations of wages, labor productivity and value added in the production sector (*middle left*), so that domestic prices and total consumers' expenditure are recalculated at the end of a cycle of calculations. Finally, because labor skills are as important as capital equipment, the fourth circuit (*black arrows*) sets forth interrelated factors of education and training that interact both with the total labor force (in terms of productivity and demands for skills) and with consumption demands (in terms of expenditures on education).


DOMESTIC PRICES
TOTAL CONSUMERS' EXPENDITURE
DOMESTIC PRICES
TOTAL CONSUMERS' EXPENDITURE
IMPORTS
EXPORTS
FOREIGN PRICES
BALANCE OF TRADE
RATES OF GROWTH IN EXPORTS
OUTPUT LEVELS
INVESTMENT DEMANDS
VALUES ADDED PER UNIT OF OUTPUT
AVERAGE WAGE RATE
RATES OF GROWTH IN DEMANDS
CONSUMPTION DEMANDS
INDUSTRIAL DISTRIBUTION OF LABOR
INDUSTRIAL DISTRIBUTION OF ASSETS
TOTAL LABOR FORCE
DEMAND FOR SKILLS
PAST EDUCATIONAL DISTRIBUTION OF POPULATION
EDUCATION AND TRAINING
ERROR
SUPPLY OF SKILLS
EDUCATIONAL DISTRIBUTION OF LABOR
EDUCATIONAL DISTRIBUTION OF POPULATION

a

AGE	FEMALES (1940)	SURVIVING FEMALES (1955)	SURVIVING DAUGHTERS (1940–1954)
0–14	14,459	16,428	4,651
15–29	15,264	14,258	10,403
30–44	11,346	14,837	1,374

b

AGE	AGE 0–14	AGE 0–14	AGE 15–29	AGE 30–44
0–14	14,459 (A)	4,651 / A	10,403 / B	1,374 / C
15–29	15,264 (B)	14,258 / A	0	0
30–44	11,346 (C)	0	14,837 / B	0

FUTURE POPULATIONS can be calculated by the use of matrix algebra. In this example it is assumed that the pattern of births and deaths remains constant over a 45-year period, and the calculation is confined to the future composition of the female population contained in three broad age-groups. The figures are samples and projections based on the 1940 U.S. Census; arrayed in tabular form (*a*) they show in three successive columns the numbers within each of three female age-groups in 1940, the numbers in each of these

the mathematical model we will take up here, however, the children are assigned according to the age-group of the wife alone.

In this kind of analysis of the process of change the changes from one time period to the next are most conveniently presented in a square array, or matrix [*see illustration at top of these two pages*]. The first step is to arrange the information about the age composition of the population in a single column of figures. Next, a square grid of rows and columns—the matrix proper—is constructed, with one row and one column for each age-group in the adjacent population column. Across the top row of the matrix are entered the number of children born to mothers in each age-group during the chosen time interval. Then the number of individuals in each age-group that are alive at the end of the chosen time interval, and have thus advanced from one age-group to the next, are entered diagonally on the grid, from top left to bottom right, in the spaces just below the main diagonal.

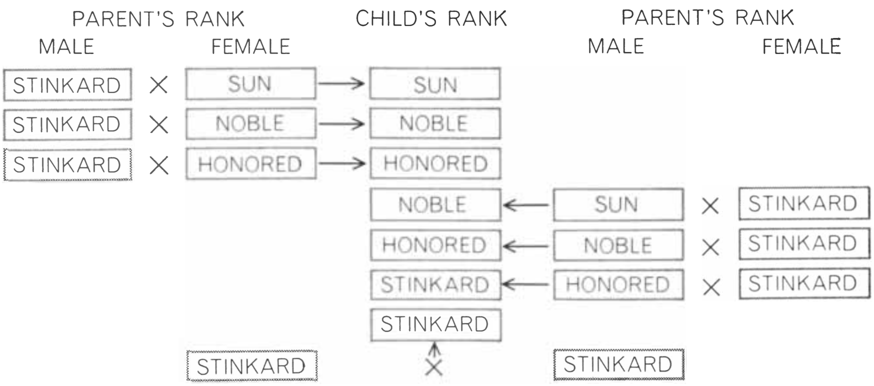
The information is now ready to be manipulated. By dividing the number of newborn individuals and the number of surviving individuals shown in each column of the matrix by the total number of people in the adjacent age-group column, a coefficient of birthrates and survival rates is obtained; these coefficients can be arrayed in a second matrix. If the figures in the age-group column are now multiplied by the coefficient matrix, the resulting totals in each age-group will show the estimated composition of the population at the end of the chosen time interval. Assume that this has been an interval of x years; if an estimate of population composition at the end of $2x$ years is desired, the original age-group figures are multiplied by the coefficient matrix raised to the power of 2. If a $3x$-year projection is desired, raise the multiplier to the power of 3, and so forth. In this example the manipulation of figures has been simply called multiplication. Mathematicians will realize that the actual process is premultiplication of the column by the coefficient matrix. The reason is that the commutative law of arithmetic does not apply to matrix multiplication [see "Number," page 50].

Projections of this kind assume that the pattern of births and survivals remains unchanged throughout the period of the estimate; this is another example of the differences between models and reality. Nonetheless, even the unreality is illuminating. Manipulation of this rigid model demonstrates that, if a human population actually achieved a fixed pattern of births and deaths, it would eventually attain a stable age composition and a steady rate of growth.

In actuality, of course, an unchanging matrix of coefficients is an extreme improbability: the birthrate and death rate are subject to chance or to systematic influences or to both. The investigator therefore works toward an approximation of reality by manipulating the values in the coefficient matrix, a process known as complicating the model. It is not difficult to work out conditions in which a population would tend to attain an upper or a lower limit, and thus a demographer's device can be extended to the study of problems in such fields as ecology and epidemiology.

Another point is worth reemphasis: the very process of mathematization helps to demonstrate that a certain unity exists in the structure of many apparently different problems. The population-projection matrix described here has been presented in a demographic context. A precisely similar projection can be made for populations of inanimate objects, such as telegraph poles, railroad cars or apartment buildings, and the model can then be used to study problems of industrial inventory or urban renewal. In such cases investment rates are substituted for birthrates and obsolescence for death. Whatever the age composition of the original stock, the model will show the replacements (and extensions) needed to keep the stock on any selected time path.

Another kind of structure, the investigation of which is the delight of ethnologists, is kinship—with its associated complex patterns of marriage and descent, clans, phratries and tribes and eventually considerations of class structure and social mobility. In such studies difference equations often provide a



MARITAL CUSTOM among the Natchez Indians of North America required that one partner in each match be a proletarian Stinkard. The table lists the seven possible unions and

m_1 (TIME T, 1940, 1955)

$$\text{TIME } T+15 \begin{pmatrix} .32167 & .68154 & .12110 \\ .98610 & 0 & 0 \\ 0 & .97203 & 0 \end{pmatrix} \times \begin{pmatrix} 14{,}459 \\ 15{,}264 \\ 11{,}346 \end{pmatrix} = \begin{pmatrix} 16{,}428 \\ 14.258 \\ 14{,}837 \end{pmatrix}$$

d

m_2 (TIME T, 1940, 1970)

$$\text{TIME } T+30 \begin{pmatrix} .77554 & .33694 & .03895 \\ .31719 & .67207 & .11942 \\ .95852 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 14{,}459 \\ 15{,}264 \\ 11{,}346 \end{pmatrix} = \begin{pmatrix} 16{,}799 \\ 16{,}200 \\ 13{,}863 \end{pmatrix}$$

e

m_3 (TIME T, 1940, 1985)

$$\text{TIME } T+45 \begin{pmatrix} .58172 & 56643 & 09392 \\ .76476 & .33226 & .03841 \\ .30832 & .65327 & .11608 \end{pmatrix} \times \begin{pmatrix} 14{,}459 \\ 15{,}264 \\ 11{,}346 \end{pmatrix} = \begin{pmatrix} 18{,}123 \\ 16{,}565 \\ 15{,}747 \end{pmatrix}$$

groups still alive in 1955 and finally the number of daughters born to each group during the same period. These figures are next arrayed in a square matrix (*b*) and then converted into a matrix of coefficients (*c*). When the coefficient matrix and the 1940 age-groups are now multiplied, the result (*the column of numbers under 1955 at "c"*) is the predicted female population for that year. A multiplication of the square and cube of the coefficient matrix, in turn (*"d" and "e"*) yields predictions for the years 1970 and 1985.

convenient way to manipulate the data. A classic example of social stratification reportedly existed among the Natchez, an American Indian tribe of the Mississippi basin. Fundamentally there were two classes among the Natchez: the aristocrats and the proletarians. The aristocrats were subdivided into three ranks, with Suns at the top, Nobles in the middle and Honoreds—still aristocrats but only a cut above the proletarian Stinkards—at the bottom.

As European travelers described the Natchez society, a kind of continuous social ferment was built into it by the marriage rules. At least one partner in any marriage—husband or wife—was always drawn from the Stinkard class. At the same time a child born into the aristocracy had the rank of its mother. Thus the child of a Sun, a Noble or an Honored mother and a Stinkard father was a Sun, a Noble or an Honored. The child of a Stinkard mother and a Sun father, however, inherited a rank one level below his father's and was only a Noble. This downgrading was progressive, so that the issue of a Stinkard mother and a Noble father was an Honored, and of a Stinkard mother and an Honored father a common Stinkard. At the bottom of the social heap Stinkard and Stinkard produced Stinkard [*see illustration at bottom of these two pages*].

The Natchez tribe has long since disappeared, but students of class structure have continued to wonder if such a complex social system could possess any natural stability. To mathematize the problem it is necessary to assume that the population is stable, that each Natchez individual married once and only once and that each marriage produced one boy and one girl. Using these values in a mathematical model, it is soon evident that a stable class structure is impossible unless Suns and Nobles are initially absent from the population. Otherwise a few generations produce such a numerous aristocracy that there are no longer enough proletarian marriage partners to go around.

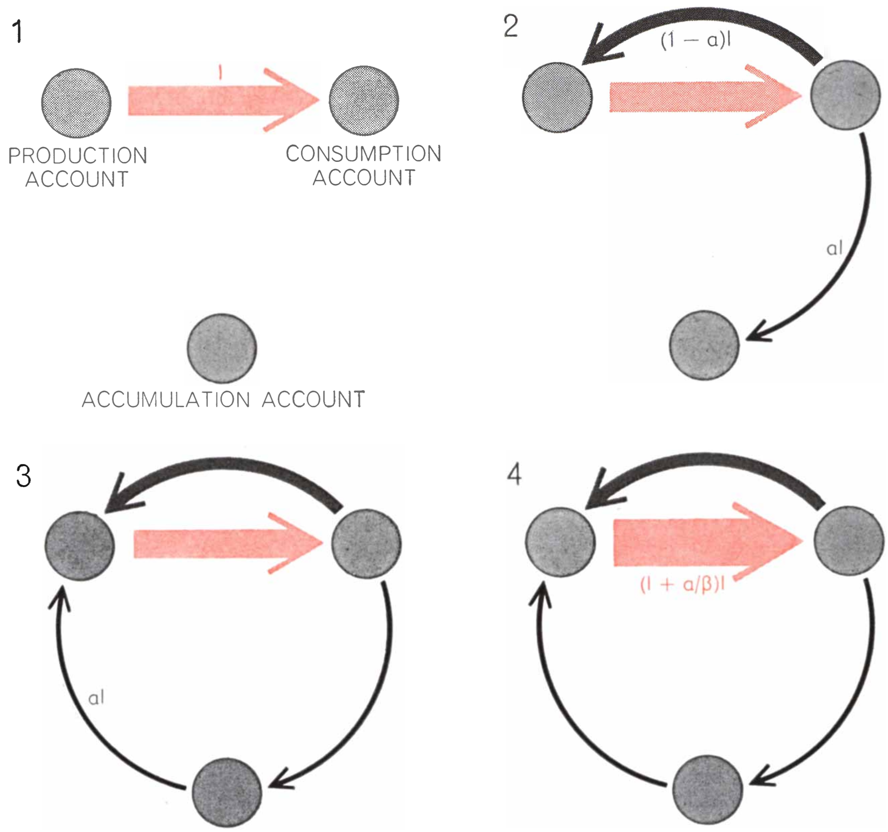
The behavior of the model suggests that the real Natchez did not in fact have a stable class structure. The inadequacy of the actual observations, however, makes it impossible to be sure. Manipulation of the model to allow for possible variations in the number of marriages per individual, or the comparative productivity, say, of Sun-Stinkard v. Stinkard-Stinkard marriages, or such completely unmodeled elements as a rapidly expanding population due to a Natchez policy of conquest and assimilation, might yield significantly different results.

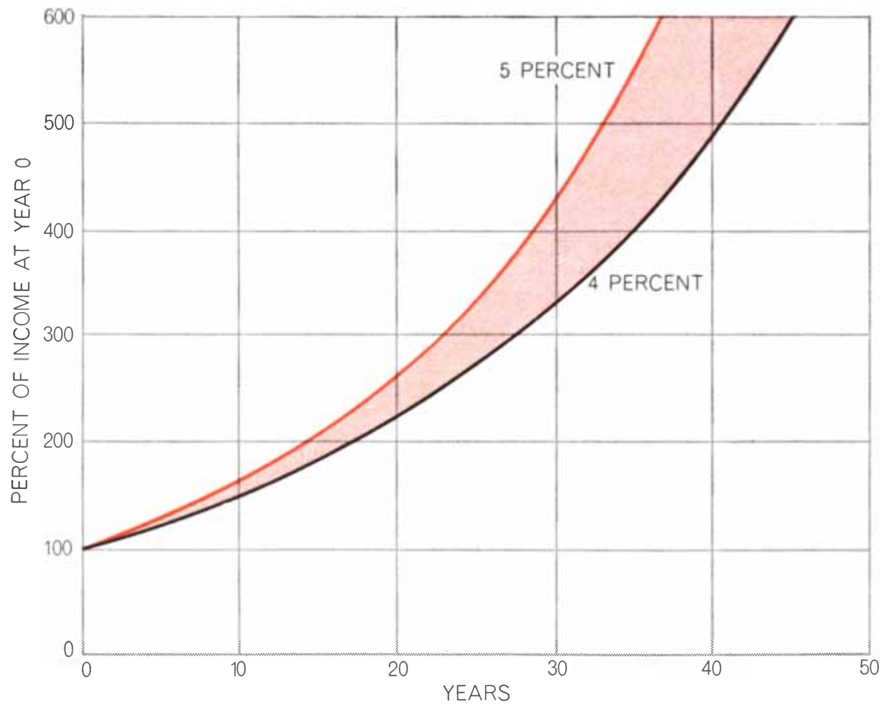
In a completely different area of study, economists who wish to examine such matters as rate of growth face the task of reducing innumerable transactions concerned with production, consumption, accumulation and foreign trade to some kind of order. One way to achieve such order is to view the economy as a vast system of interlocking accounts; in theory an enormous square matrix could be set up to do this, with each row-and-column pair representing a single account. Incoming revenue would be shown in the rows and outgoing costs in the columns. In actuality, if all the flows in an economic system were recorded in detail in a single matrix, the resulting array of rows and columns would be impossibly large and unmanageable. The solution is a selective consolidation of accounts into major classes. One such consolidation, applicable to a closed economy, reduces a complex picture to three interlocking accounts: one for production, one for consumption and one for accumulation [*see illustration on next page*].

In a model economy of this kind the

	INITIAL POPULATION	GENERATION 1	GENERATION 2	GENERATION 3	GENERATION 4	GENERATION 5
SUN	10	10	10	10	10	10
NOBLE	20	30	40	50	60	70
HONORED	40	60	90	130	180	240
TOTAL	70	100	140	190	250	320
STINKARD	500	470	430	380	320	250
TOTAL	570	570	570	570	570	570
	G	G + 1	G + 2	G + 3	G + 4	G + 5
SUN	S	S	S	S	S	S
NOBLE	N	N + S	N + 2S	N + 3S	N + 4S	N + 5S
HONORED	H	H + N	H + 2N + S	H + 3N + 3S	H + 4N + 6S	H + 5N + 10S

the rank of their issue. At first glance the system seems plausible. But mathematical analysis of a sample population (in this case containing 70 aristocrats and 500 proletarians at the start) shows a deficiency in Stinkard marriage partners after only four generations.





KEYNESIAN VIEW of a closed economy states that income equals consumption plus savings and that savings equal investment. When production, consumption and savings are interlocked in a model (*top illustration*), the growth of such a closed economy can be traced as the movement of income to the consumption account (*1*), which produces a two-way flow of outlay (*2*) into consumption spending and into savings. The second flow continues (*3*) from accumulation to production, increasing capital equipment and thereby stimulating greater production (*4, enlarged arrow*). If the rate of economic growth is to be increased, either more income must be saved and invested or greater efficiency must be achieved in the funds invested per unit of output. The importance of even slight rate increases, for example in the case of underdeveloped economies, is shown (*bottom illustration*) by comparison of a 1 percent difference between two rates of growth during a 50-year period.

production account receives money from the consumption account in exchange for sales of consumption goods and services and from the accumulation account in exchange for sales of investment goods. The production account then proceeds to pay out to the consumption account the value of its sales in the form of income. This income is the only revenue the consumption account receives; within that account the revenue is divided into two unequal parts. The larger part is spent on additional consumption goods and services (with a resulting flow back into the production account) and the rest goes into savings (with a resulting flow back into the accumulation account). To close the system the accumulation account then pays these savings into the production account in return for further investment goods.

Two of the assumptions in this model are obvious. The first is that total income is exactly equal to the sum expended on consumption goods and savings. The second is that savings, in turn, are exactly equal to expenditures on investment goods. Two other assumptions, although less immediately apparent, prove on examination to be the key relations in this closed-economy model. One of these is that the economy's inhabitants save a fixed proportion of their income (here called α). The other is that a fixed coefficient of proportionality exists between expenditures on investment goods, to wit additional capital, and increased yields from the production account, to wit additional production. This coefficient is here called β.

Any closed economy that is based on these assumptions can only grow at the constant rate represented by α/β. If it is to increase its rate of growth, the system must either save and invest a larger proportion of its income (that is, increase α) or contrive to use less capital per unit of output (that is, reduce β) or both. As an example, let these relations be quantified as follows: α is a savings rate of 10 percent of income and β is a capital-to-output ratio of 2.5. Under these circumstances α/β will yield a growth rate of 4 percent. But if the value of α can be increased to 12.5 percent, or if the value of β can be reduced to 2 percent, the economy's rate of growth will rise from 4 percent to 5.

A final example of the use of mathematics in describing and analyzing social situations can be found in the area of education. Concern over an adequate future supply of teachers, for in-

"Who is to bell the cat?
It is easy to propose
impossible remedies."
Aesop

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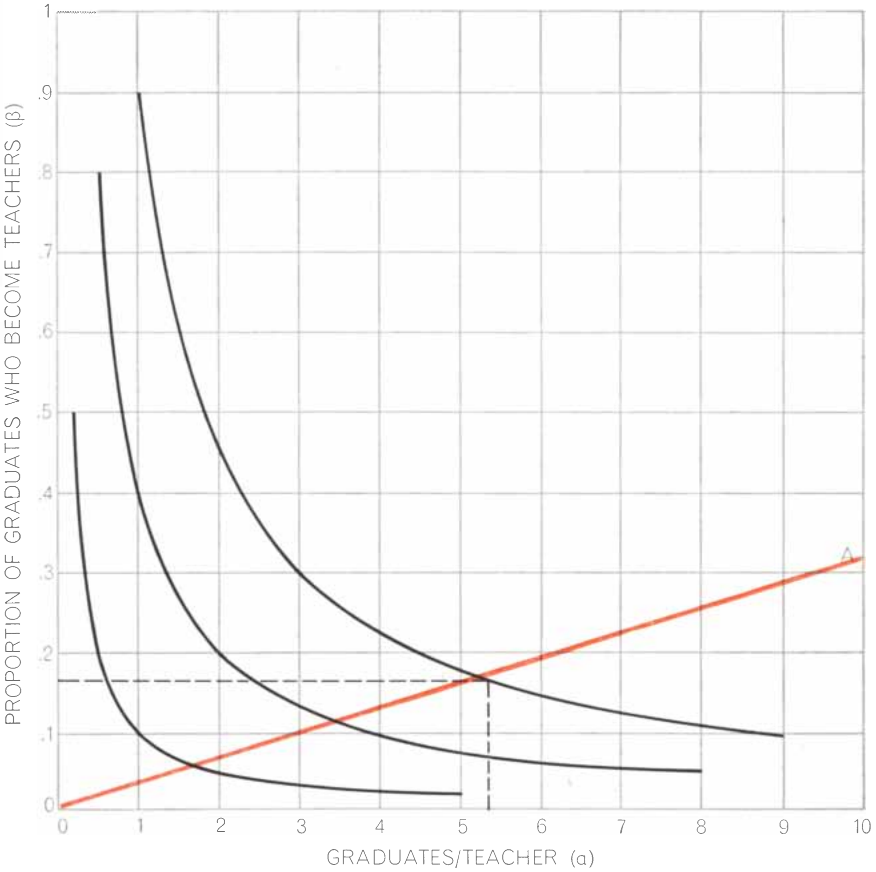

GEOMETRIC DEMONSTRATION of the problems involved in a future supply of teachers, as a first step, shows that all the possible rates of growth for the stockpile of teachers are interactions between the number of students each teacher can bring to the graduate level and the proportion of graduates who then become teachers. For each growth rate a separate hyperbolic curve (*black*) will plot all the possible combinations of these related factors. A separate calculation, in turn, establishes a linear relation between these same two factors in response to the varying stimuli of higher and lower rates of pay for teachers. When this linear relation is inserted in the graph (*straight colored line*), its intersections with the family of hyperbolas pinpoint specific values for each of the two factors in each instance. It is then possible to make another calculation that will show the corresponding rates of pay.

stance, requires analysis of the way teachers and graduates interact. In the simplest kind of case it can be taken as self-evident that the number of graduates produced by any educational system is in some way proportional to the number of teachers engaged in the system. An increase in the number of teachers, in turn, depends on the proportion (supposedly constant) of graduates of the system who become teachers and the proportion (also supposedly constant) of the existing teacher pool that is lost to the profession through death, retirement or resignation.

The ratio of graduates produced per teacher, in this example, may be designated α, the proportion of graduates who become teachers designated β and the lost-to-the-profession proportion designated γ. It is then easy to devise an equation for the net growth of teachers and graduates (designated δ). The equation is $\delta = \alpha\beta - \gamma$. At first glance it seems that nothing but the obvious has been mathematized. All the equation says is that if more teachers are wanted, it is necessary either (1) to arrange for the existing number of teachers to turn out a greater number of graduates (that is, increase α), or (2) to persuade a greater number of graduates to enter the teaching profession (that is, increase β), or (3) to diminish the wastage rate in the existing stock of teachers (that is, reduce γ), or finally (4) to do all three at once.

Actually more than the obvious has been achieved by reducing the problem to mathematical relations. When these various interacting considerations are expressed geometrically [*see illustration above*], it becomes possible to examine still other interactions. Suppose the social situation is such that little or nothing can be done to reduce teacher wastage; no board of education, say, will reemploy retired teachers. Such a

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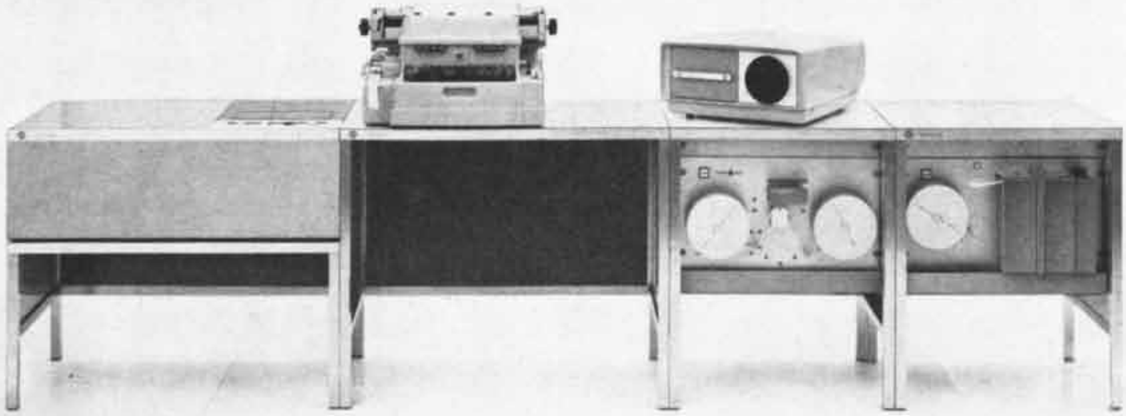
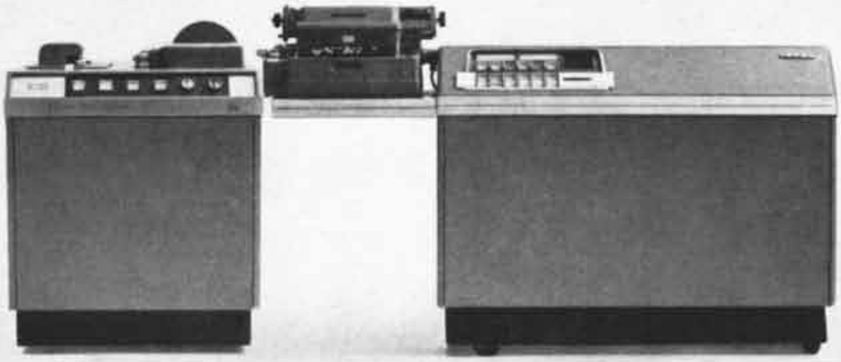


LGP-21



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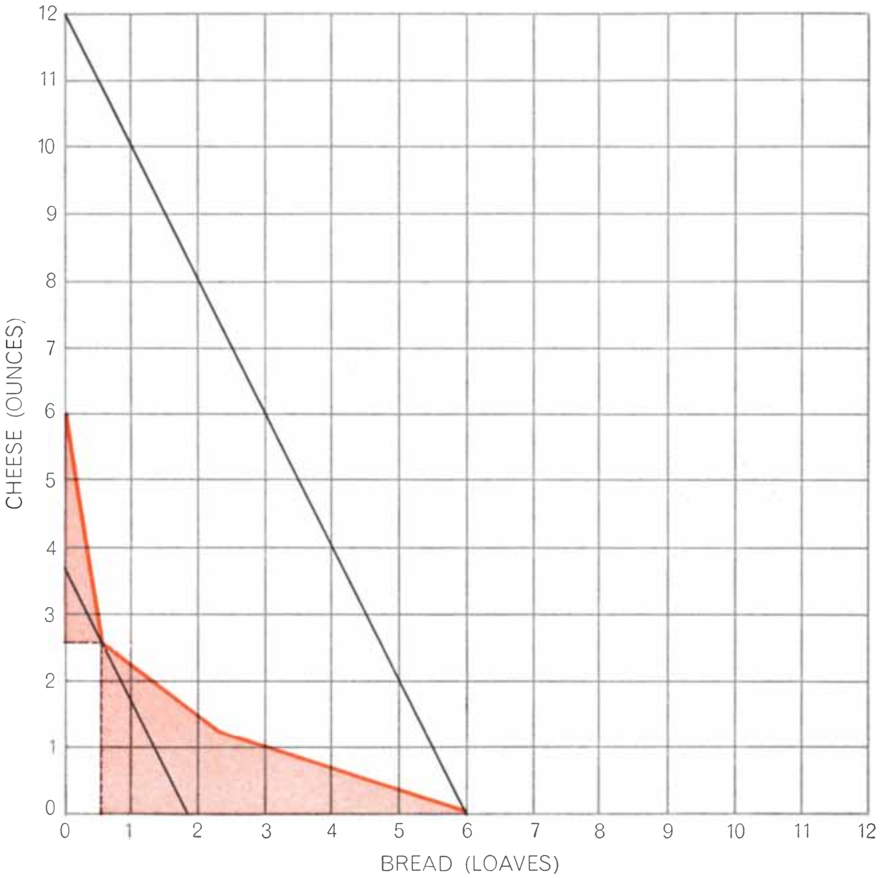
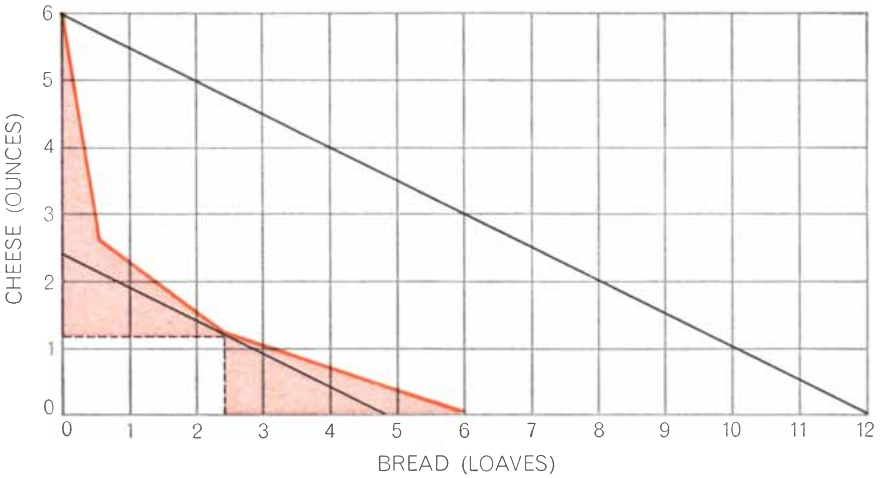
circumstance prevents any significant manipulation of γ, but courses of action are still possible with regard to α and β. What might be achieved by raising teachers' wages? Such an action would certainly bring more graduates into the teaching profession (thus affecting β) and might also improve morale and increase teacher efficiency (thus affecting α).

Indeed, it can be imagined that both α and β are related hyperbolically to pay; that is, at very low rates of pay the teachers would do no work and no graduates would enter the profession, whereas at very high rates both the teachers' efficiency and the number of graduate careerists would approach upper limits. In such a case the relation between α and β is linear, and this linear relation, when it is inserted in the geometric construction, enables corresponding rates of pay to be calculated for various values of α and β.

All the preceding examples are concerned with describing the world as it is without considering the mechanisms that determine how it comes to be that way. This latter consideration introduces the area of decisions. To a large extent the maintenance and modification of social patterns depend on innumerable private and public decisions consciously undertaken in the hope of achieving certain ends, and therefore the study of the decision process is an essential part of the social sciences.





MINIMUM-COST DIET demonstrates that even decisions made under conditions of certainty are not always palatable. The example assumes that three required nutrient factors are present in varying proportions in both bread and cheese (two ounces of cheese, in this case, contain as much protein as six loaves of bread do). Plotting the three proportions on a graph produces a convex boundary (*solid colored line*), any point on which satisfies all constraints regarding nutrient demands. Now the least expensive combination of the two foodstuffs must be found. If 12 ounces of cheese cost the same as six loaves of bread (*upper graph*), the minimum-cost diet is reasonable: more than two ounces of cheese go with every half-loaf of bread. But if the price ratio is reversed (*lower graph*), the minimum-cost diet proves to contain an unacceptable bulk of cheap bread in proportion to expensive cheese.

To a large extent the decision process can be formulated and analyzed mathematically. In addition to the classical techniques of mathematics today's decision-maker can draw on such new methods as linear programming, game theory and statistical decision theory. As a preliminary, however, it is necessary to distinguish between decisions made under conditions of certainty and those made under conditions of uncertainty. And within these two categories it is necessary to distinguish between decisions that relate to a single period and those that involve a course of action over several periods.

An example of the first category and subcategory—a single-stage decision taken under conditions of certainty—can be found in the dietitian's problem of devising an adequate diet at minimum cost. This problem defines an adequate diet in terms of certain minimum quantities of such nutritional factors as proteins, carbohydrates and fats, vitamins, caloric content and the like. Various foods, with fixed prices, contain these nutritional elements in known quantities. The question is: How much of each foodstuff should be bought in order to meet the nutritional requirement at minimum cost?

Because the function to be minimized in this problem is linear, the solution requires the technique of linear programming. If a simplified case—involving only two foodstuffs and three nutritional elements—is taken, the problem can also be interpreted geometrically as a construction of two axes, one for each foodstuff [*see illustration at left*]. With n foods the same geometric construction

can be thought of as repeated in n dimensions; as n increases, the number of calculations needed to find a solution also increases.

In practice the solution of the minimum-cost diet problem usually turns out to involve very few foods and to offer a far from acceptable diet. The reason for this is that the problem is usually too narrowly formulated; for example, the bulk of the food that constitutes the minimum-cost diet is not taken into account, and this bulk often turns out to be very large. As has recently been found, if a limit is put on the weight of the food to be consumed each day, a much more varied diet emerges with no appreciable increase in cost. This problem of narrow formulation illustrates a feature that is characteristic of all complex calculations: mathematical methods are literal methods. They solve problems as they are formulated; it is up to the investigator to see that they are formulated sensibly.

In any decision made under conditions of uncertainty, what was previously a known (or supposedly known) magnitude has been replaced by an entire distribution of magnitudes. The

			PRODUCTION ACCOUNTS														INCOME AND OUTLAY ACCOUNTS
			COMMODITIES				INDUSTRIES				CONSUMERS' GOODS AND SERVICES			GOVERNMENT PURPOSES			INDIRECT TAXES AND SUBSIDIES
			FUEL AND POWER	METALS, ENGINEERING, CONSTRUCTION	AGRICULTURE, MANUFACTURING	SERVICES	FUEL AND POWER	METALS, ENGINEERING, CONSTRUCTION	AGRICULTURE, MANUFACTURING	SERVICES	FOOD, DRINK, TOBACCO	CLOTHING AND HOUSEHOLD	OTHER	DEFENSE	HEALTH, EDUCATION, CHILD CARE	OTHER	INDIRECT TAXES
			1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
PRODUCTION ACCOUNTS: COMMODITIES	FUEL AND POWER	1	0	0	0	0	590	353	274	476	0	647	94	74	46	41	0
	METALS, ENGINEERING, CONSTRUCTION	2	0	0	0	0	235	6,077	750	756	0	373	88	637	62	98	0
	AGRICULTURE, MANUFACTURING	3	0	0	0	0	72	693	4,032	882	4,046	1,363	381	28	147	39	0
	SERVICES	4	0	0	0	0	281	1,724	1,767	367	1,478	878	2,569	23	266	67	0
INDUSTRIES	FUEL AND POWER	5	2,686	4	12	0	0	0	0	0	0	0	0	0	0	0	0
	METALS, ENGINEERING, CONSTRUCTION	6	5	15,349	74	17	0	0	0	0	0	0	0	0	0	0	0
	AGRICULTURE, MANUFACTURING	7	12	43	11,769	12	0	0	0	0	0	0	0	0	0	0	0
	SERVICES	8	0	25	4	10,640	0	0	0	0	0	0	0	0	0	0	0
CONSUMERS' GOODS AND SERVICES	FOOD, DRINK, TOBACCO	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	CLOTHING AND HOUSEHOLD	10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	OTHER	11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
GOVERNMENT PURPOSES	DEFENSE	12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	HEALTH, EDUCATION, CHILD CARE	13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	OTHER	14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
INCOME AND OUTLAY ACCOUNTS: INDIRECT TAXES AND SUBSIDIES	INDIRECT TAXES	15	1	49	85	0	53	76	126	604	1,221	471	372	10	20	14	0
	LESS SUBSIDIES	16	0	0	0	0	−3	0	−257	−111	0	−118	0	0	0	0	0
INSTITUTIONAL SECTORS	DISTRIBUTION OF PROPERTY INCOME	17	0	0	0	0	101	1,401	1,369	1,899	0	524	0	0	0	0	0
	PRIVATE SECTOR	18	0	0	0	0	792	4,341	2,408	5,028	0	86	118	718	921	699	0
	PUBLIC SECTOR	19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3,405
CAPITAL-TRANSACTION ACCOUNTS: COMMODITIES	FUEL AND POWER	20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	METALS, ENGINEERING, CONSTRUCTION	21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	AGRICULTURE, MANUFACTURING	22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
INDUSTRIES, REPLACEMENTS	FUEL AND POWER	23	0	0	0	0	185	0	0	0	0	0	0	0	0	0	0
	METALS, ENGINEERING, CONSTRUCTION	24	0	0	0	0	0	184	0	0	0	0	0	0	0	0	0
	AGRICULTURE, MANUFACTURING	25	0	0	0	0	0	0	312	0	0	0	0	0	0	0	0
	SERVICES	26	0	0	0	0	0	0	0	416	0	0	0	0	0	0	0
INDUSTRIES, EXTENSIONS	FUEL AND POWER	27	0	0	0	0	70	0	0	0	0	0	0	0	0	0	0
	METALS, ENGINEERING, CONSTRUCTION	28	0	0	0	0	0	119	0	0	0	0	0	0	0	0	0
	AGRICULTURE, MANUFACTURING	29	0	0	0	0	0	0	91	0	0	0	0	0	0	0	0
	SERVICES	30	0	0	0	0	0	0	0	209	0	0	0	0	0	0	0
CONSUMERS, REPLACEMENTS	DWELLINGS	31	0	0	0	0	0	0	0	0	0	132	0	0	0	0	0
	DURABLES	32	0	0	0	0	0	0	0	0	0	227	337	0	0	0	0
CONSUMERS, EXTENSIONS	DWELLINGS	33	0	0	0	0	0	0	0	0	0	164	0	0	0	0	0
	DURABLES	34	0	0	0	0	0	0	0	0	0	171	234	0	0	0	0
GOVT. REPLACEMENTS	ALL SOCIAL CAPITAL	35	0	0	0	0	0	0	0	0	0	0	0	1	42	20	0
GOVT. EXTENSIONS	ALL SOCIAL CAPITAL	36	0	0	0	0	0	0	0	0	0	0	0	0	44	26	0
INSTITUTIONAL SECTORS	CHANGES IN ASSETS AND CLAIMS	37	0	0	0	0	−7	−83	−81	−112	0	0	0	0	0	0	0
	PRIVATE SECTOR	38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	PUBLIC SECTOR	39	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
REST OF THE WORLD ACCOUNT		40	145	665	1,308	347	333	560	1,045	255	296	0	277	120	0	26	0
TOTAL OUTGOINGS			2,849	16,135	13,252	11,016	2,702	15,445	11,836	10,669	7,041	4,918	4,470	1,611	1,548	1,030	3,405

MATRIX MODEL of the British economy presents the accruals and disbursements for the year 1960 within the four national accounts. These four are here subdivided into 40 groups; in the author's larger model there are 253 such subdivisions. The accounting nature of the model is shown (*color*) for entry No. 10, a consumers' subdivision of the "Production accounts" that summarizes clothing and household transactions. At the intersection of row 10 and column 18 is entered the total amount (4,918 million pounds) spent by consumers on clothing and household items. Column 10, in turn, presents the component parts of this expenditure.

problem is to find out the nature of this distribution and decide how it should influence the decision. Questions of this kind involve the fields of probability and statistics. A classic example is the tossing of heads and tails with a "biased" coin. Under normal circumstances the probability of an ordinary coin's falling heads up at a single throw is, for all practical purposes, 1/2, and the probability of two consecutive heads-up falls, in turn, is 1/4. Obviously a trick coin with two heads would not yield the same probability: the single-throw prospect for heads would improve from 1/2 to 1. Imagine, in place of this brutal kind of bias, a coin so ingeniously contrived that the probability of obtaining a head at a single throw ranges evenly between the limits of 0 and 1. As Pierre Simon de Laplace established with his law of succession, the chance of obtaining a head at each of two throws with such a coin is not 1/4 but 1/3.

What has happened, of course, is that a different distribution of magnitudes has been substituted for the usual distribution. In the real world no such coins exist; however, the real world abounds with opportunities for deci-

CAPITAL-TRANSACTION ACCOUNTS

…MMODITIES: …Y INCOME, PRIVATE SECTOR, PUBLIC SECTOR (16–19) · INDUSTRIES, REPLACEMENTS: METALS, ENGINEERING, CONSTRUCTION; AGRICULTURE, MANUFACTURING; FUEL AND POWER (20–22) · INDUSTRIES, EXTENSIONS: METALS, ENGINEERING, CONSTRUCTION; AGRICULTURE, MANUFACTURING; FUEL AND POWER; SERVICES (23–26) · CONSUMERS, REPLACEMENTS: METALS, ENGINEERING, CONSTRUCTION; AGRICULTURE, MANUFACTURING; FUEL AND POWER; SERVICES (27–30) · CONSUMERS, EXTENSIONS: DWELLINGS; DURABLES (31–32) · GOVT. REP.: DWELLINGS; DURABLES (33–34) · GOVT. EXT.: ALL SOCIAL CAPITAL (35) · INSTITUTIONAL SECTORS: ALL SOCIAL CAPITAL (36) · CHANGES IN ASSETS AND CLAIMS: PRIVATE SECTOR; PUBLIC SECTOR (37–39) · REST OF THE WORLD ACCOUNT (40) · TOTAL INCOMINGS

16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	TOTAL INCOMINGS
0	0	0	0	−6	0	0	38	0	0	0	52	0	0	0	0	0	0	0	0	0	0	0	0	170	2,849
0	0	0	0	0	339	0	142	174	289	331	262	430	298	737	107	262	646	410	63	274	0	0	0	2,295	16,135
0	0	0	0	0	0	230	0	0	6	0	0	0	3	0	0	33	0	47	0	0	0	0	0	1 250	13,252
0	0	0	0	0	0	0	5	4	11	65	1	2	0	60	25	169	0	217	0	1	0	0	0	1,036	11,016
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2,702
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	15,445
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	11,836
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	10,669
0	0	7,041	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	7,041
0	0	4,918	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4,918
0	0	4,255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	215	4,470
0	0	0	1,611	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1,611
0	0	0	1,548	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1,548
0	0	0	1,030	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1,030
0	0	0	0	0	0	1	0	6	6	20	1	7	8	29	0	100	0	125	0	0	0	0	0	0	3,405
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−489
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	179	5,473
0	6,139	0	1,652	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	22,900
489	−066	3,624	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5,874
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−6	0	0	0	−6
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	357	0	0	0	357
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	240	0	0	0	240
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	185
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	184
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	312
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	416
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	246	0	0	0	316
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	320	0	0	0	439
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	218	0	0	0	309
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	617	0	0	0	826
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	132
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	564
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	293	189	0	646
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	394	0	0	799
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	63
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	205	0	275
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2,213	−282	0	1,648
0	0	3,062	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	74	0	3,136
0	0	0	−51	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	236	0	1	186
0	0	0	86	0	18	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−344	0	0	0	5,146
489	5,473	22,900	5,874	−6	357	240	185	184	312	416	316	439	309	826	132	564	646	799	63	275	1,648	3,136	186	5,146	

First entry (647 million pounds) is the total spent for household heating and lighting: coal, coke, gas, electricity and oil. Negative entry at row 16 (−118 million pounds) represents housing subsidies paid by government. Another entry, at row 18 (86 million pounds), is the total paid in salaries to domestic servants during 1960. Author's larger model deals in more detail not only with the different branches of industry (a 31 × 31 square matrix stands in place of the 4 × 4 square matrix shown here at the intersection of rows 1, 2, 3 and 4 and columns 5, 6, 7 and 8) but also with the consumption, accumulation and foreign transactions of the economy.

sions under conditions of uncertainty where a failure to examine the assumptions about the kind of distribution proves to be quite as hazardous as betting blindly against a biased coin.

The combination of theories and facts in the social sciences will produce useful models of real situations. If a model, in turn, is combined with a set of aims, the product that results is a policy. It is because of this, I think, that large quantitative models of national economic systems, intended for practical use, are at present being built in many countries. An economy can be viewed as a system that transforms information into decisions. Today a quantitative economic model, if it is made detailed enough and reliable enough, should be able to provide much useful information for decision-makers.

For the past four years my colleagues and I at the University of Cambridge have been working on such a device, a computable model of the British economy. We now have a working prototype which, although still much in need of improvement, shows that the task is a manageable one. As a first step we constructed a model for a steady state of economic growth starting from a future year that for the moment we have arbitrarily set as 1970 [*see illustration on page 169*]. This model is based on specific assumptions about the British standard of living in 1970 and about the economy's rate of growth thereafter. The economic consequences of these assumptions are set out as a series of balances. First are the balances of supply and demand for the 31 product groups distinguished in our model. Second are the balances of revenues and costs in the corresponding branches of production. Three others complete the series: the balance of supply and demand for labor, the balance of saving and investment and the balance of the external account (Britain's foreign trade).

The variables in the model are the entries in a square matrix consisting of 253 pairs of rows and columns (each of which represents a balancing account, showing the incomings and outgoings of some branch or sector of the economy), and the prices and quantities associated with these entries [see "Input-Output Economics," by Wassily W. Leontief; SCIENTIFIC AMERICAN, October, 1951]. The entire series of entries we call the "social-accounting matrix," or SAM; a summary presentation of SAM's row-and-column entries appears on the preceding two pages. Working within such a framework at least assures us results that are consistent in an arithmetic and accounting sense.

Operating the model involves a computer run. The input required to obtain a set of output balances consists of 5,000 to 6,000 numerical statements; the run involves about 30 million multiplications. The Atlas computer we use needs only 22 seconds to do these multiplications. Indeed, only the development of electronic computers has made macromodels of this kind possible. A generation ago, when the discipline of econometrics was coming into being with the help of desk calculators, such models would have been unthinkable. Twenty-two seconds of work for Atlas is the equivalent of 60 man-years with a desk calculator.

Now that SAM is operating, although much refinement lies ahead, we have begun work on our second task. This is: Given the present state of the British economy, what are the changes that are needed to bring tomorrow's economy, so to speak, more in line with our projection for 1970—the day after tomorrow? Eventually we hope to make these two parts of the model interact.

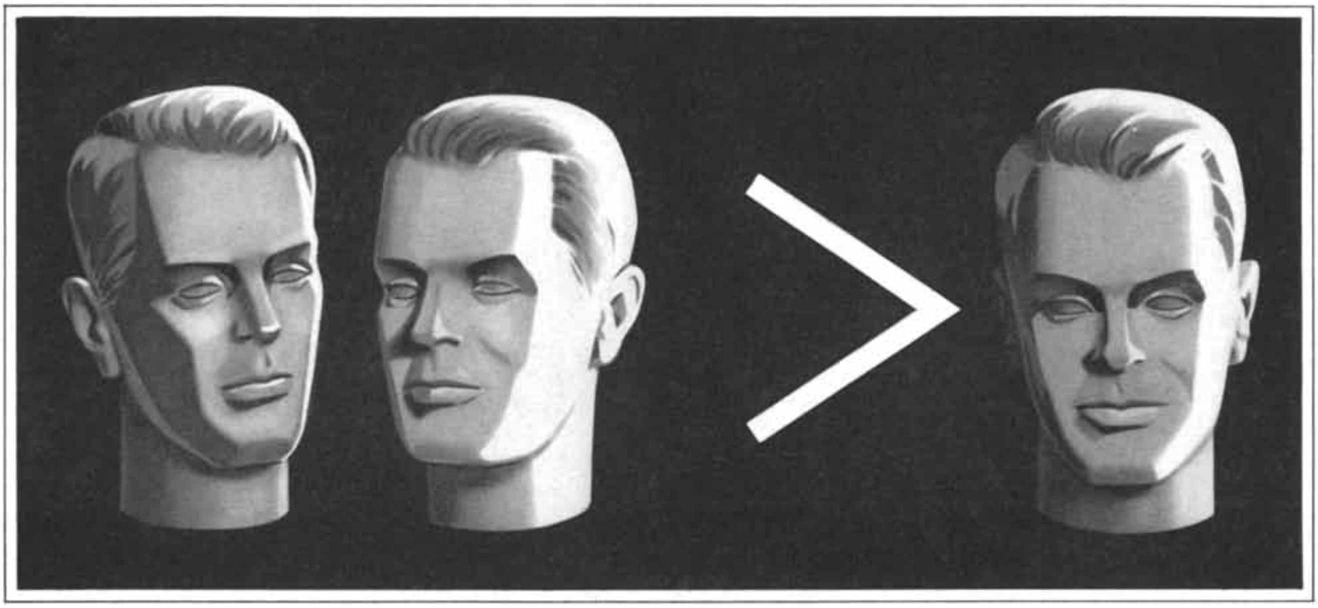
In this connection, one of the main advantages of economic models is that they can be made to work out the implications of any assumptions we choose to make. We can, for example, establish a "closed" system and observe how this will develop "naturally" in the future; given an initial set of values, the model will trace out the future paths of a number of variables. In moving from today's facts toward 1970's assumptions, however, our interest frequently centers on ways to bring about a state of affairs that does not seem to happen naturally. There are many such problem sectors: how to reduce unemployment, say, or how to increase the proportion of college and postcollege graduates in the population, or how to avoid recurrent balance-of-payment crises. For each of these cases we must open our model at some point and introduce into it a new feature: a precise statement of aims. The model will then show how the system balances out on the assumption that these aims are met. Meanwhile, as we increase the size of our original model, we have to decide whether it is more practical just to add categories or to set up submodels instead. The submodel solution seems the most promising. It is possible to imagine a decentralized system in which, for instance, the submodel for the fuel and power industries or for financial activities was maintained and operated by fuel or fi-

nancial specialists who had the necessary knowledge at their fingertips. Another advantage of the submodel method is that situations could be represented that involve changes in the individual branches of production; the fixed input-output relations that are built into a single model effectively rule out such manipulations.

Although SAM was confined to the economic aspects of British society at the start, it has shown an inevitable tendency to spread. In assessing the parts played by labor, capital and invention in producing goods and services, for example, it was necessary to consider not only different skills but also social attitudes toward research and innovation. The former has led us to the study of the education and training systems in which skills are learned, whereas the latter has led us into the field of social psychology. Before the end we may be facing a model of the entire British socioeconomic system.

In the real world such systems may not keep on an even course, either because of an inherent tendency to oscillate or because they have only a limited capacity for recovering from the succession of natural shocks to which they are inevitably subject. But like their biological and engineering counterparts, socioeconomic systems possess automatic control mechanisms (an example of this in economics is the price mechanism). Often, however, these mechanisms do not function very well, partly because they are based on limited aims and partly because they work with limited information. This is why in every country efforts are made, in greater or lesser degree, to design devices that will improve the control mechanisms.

I observed above that combining a model with a set of aims produces a policy. When a policy is combined with control systems, the combination gives rise to a plan of action. The plan in turn combines with events to give all of us our experience of socioeconomic life. This experience feeds back to modify the theories we accept, the facts we consider relevant, the aims that appeal to us and the controls we regard as efficient. By modifying these factors, experience serves to modify our models, our policies and our plans. And so it goes. We can hope that someday, thanks to the tools of mathematics, decisions may come to rest a little more on knowledge and a little less on guesswork and that the world in which we live will function a little better and be less at the mercy of unforeseen events.



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CONTROL THEORY

In technology the most advanced applications of mathematics are in the design of machines that control themselves. The same methods are relevant to the control mechanisms of living organisms and societies

by Richard Bellman

Control theory, like many other broad theories, is more a state of mind than any specific amalgam of mathematical, scientific or technological methods. The term can be defined to include any rational approach used by men to overcome the perversities of either their natural or their technological environment. The broad objective of a control theory is to make a system—any kind of system—operate in a more desirable way: to make it more reliable, more convenient or more economical. If the system is a biological one, the goal may be to understand how the system works and to reduce pain and distress.

In this article I shall mainly discuss control theories that have some explicit mathematical content, but it is clear that some of the most interesting control problems arise in fields such as economics, biology and psychology, where understanding is still notably limited. To prove that he understands, the scientist must be able to predict, and to predict he requires quantitative measurements. To make predictions that are merely qualitative, such as the prediction that an earthquake, a hurricane or a depression will occur in the near future, is not nearly so satisfying as a similar prediction associated with a specific time and place.

The ability to make a quantitative prediction is normally a prerequisite for the development of a control theory. In order to make quantitative predictions one must have a mechanism for producing numbers, and this requires a mathematical model. It might seem that the more realistic the mathematical model is, the more accurate the prediction will be and the more effective the control. Unfortunately, diminishing returns set in rapidly. The real world is so rich in detail that if one attempts to make a mathematical model too realistic, one is soon engulfed by complicated equations containing unknown quantities and unknown functions. The determination of these functions leads to even more complicated equations with even more quantities and functions—a tale without end.

The richness of the problems presented by modern civilization have led to the study of control theory on a broad front and to the development of a large variety of control systems. Although this development began long before the invention of electronic computers, the explosive growth of control theory dates from the appearance of these devices soon after World War II. For the past 20 years control theory and computers have grown side by side in an almost symbiotic relation. Without the computer most of the advanced control systems used in the military domain, in space technology and in many branches of industry could not have been developed, and without the computer they certainly could not be effectuated.

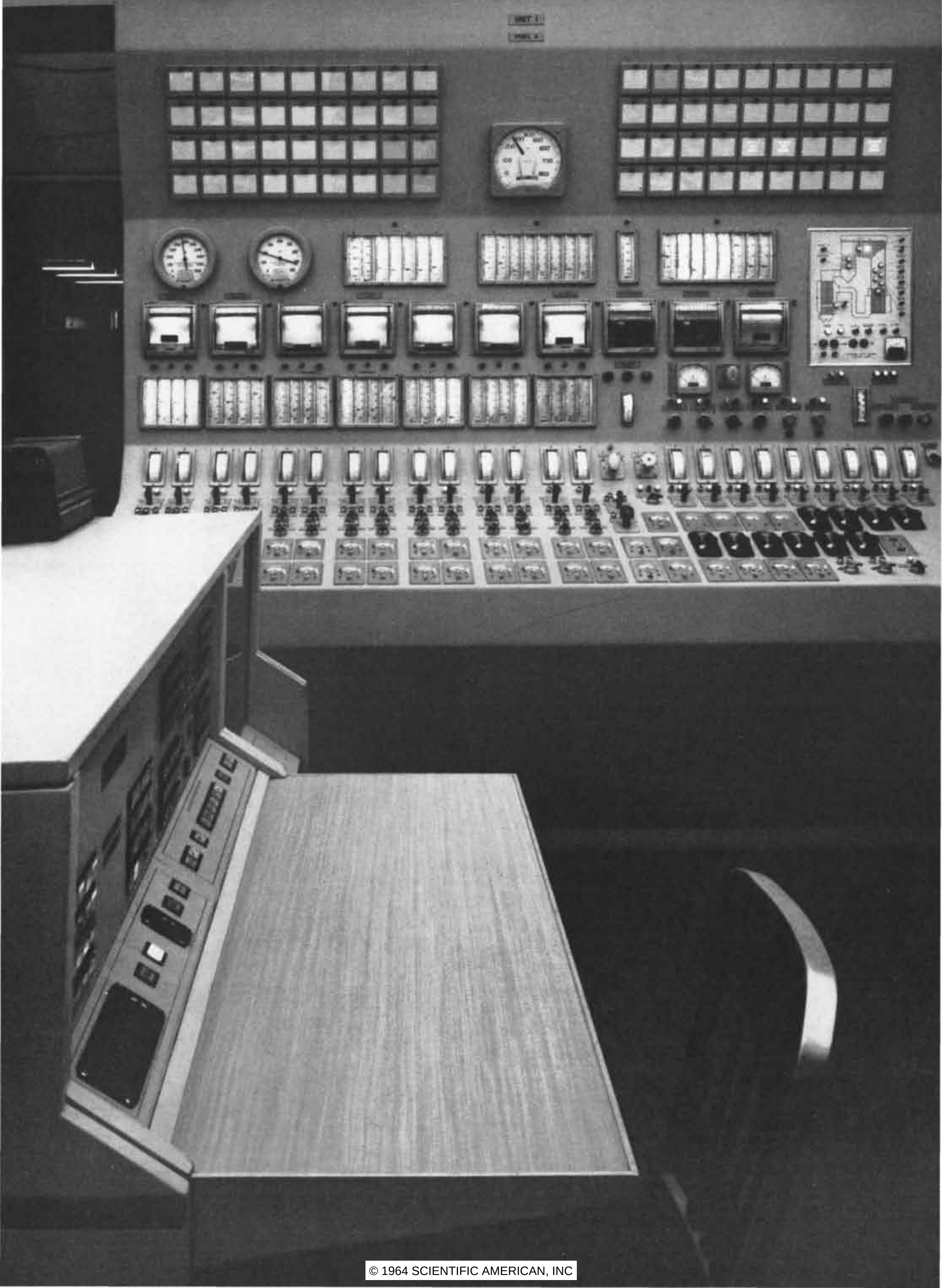
In industry, control theory, implemented by the computer, is now widely used to regulate inventories, to schedule production lines and to improve the performance of power stations, steel mills, oil refineries and chemical plants. It is estimated that about 500 computers specially designed for process control are now installed or on order. Five years ago scarcely a dozen such machines were in service.

A majority of process-control computers still operate in "open loop" fashion, which means that they monitor the process variables, analyze them in a search for possible improvements and present their recommendations to a human operator for action. In a growing number of plants, however, the loop has been closed. The decisions of the computer are directly linked to the process controls so that adjustments are made automatically.

In Rotherham, England, to select one remarkable example, a new steel plant built by Tube Investments Ltd. has been provided with three large digital computers arranged in a chain of command. The computer at the top is "off line" and is used for production planning. It receives customers' orders and classifies them according to the composition of the steel and the form of the finished product. It then calculates an efficient three-week program for the steel furnaces and rolling mills and keeps track of the program's progress. The second computer takes over when a billet has been produced by one of the furnaces and produces a full set of instructions for its subsequent treatment. This computer is "on line" and actually supervises the rolling mill. The third computer has the single task of receiving measurements of billet size and computing how they should be cut to minimize waste. This kind of integrated operation from receipt of the customer's order to final billing has become the goal of many manufacturing firms.

The industrial, military and space-flight control problems that have been presented to mathematicians for solution in the past 25 years have brought about

CLOSED-LOOP CONTROL is implemented by a computer (*console in foreground in photograph on opposite page*) in the operation of a 650-million-watt, coal-fired power plant at the Paradise Station of the Tennessee Valley Authority near Drakesboro, Ky. One of the largest power plants in the world, the station has two such computer-controlled units. In a closed-loop system the computer directly adjusts the process variables.



the revitalization of a number of moribund mathematical disciplines and have led to the creation of new theories of considerable intrinsic interest. Since control and stability are intimately related, mathematical theories devised in the 18th and 19th centuries to study such matters as the stability of the solar system have been dusted off, refurbished and applied to many problems of more current interest. These theories have included highly abstruse conceptions of the great French mathematician Henri Poincaré and the Russian mathematician A. M. Liapunov, which are now routinely employed in control studies.

The most challenging control problems encountered in science, technology, economics, medicine and even politics can be described as multistage decision processes. Traditionally they have been treated on the basis of experience, by rule of thumb and by prayerful guesses. The basic task is to determine feasible and reasonable courses of action based on partial understanding and partial information. As more information is obtained one can expect to do better, but the crux of the problem is to do something sensible *now*.

A familiar problem characterized by partial understanding is that of maintaining a healthy national economy—of avoiding a depression on the one hand and inflation on the other. A variety of regulatory devices are available for achieving the desired control. One device is to regulate the interest rate on loans. If inflationary trends develop, the interest rate is raised and money gets tighter; if a depression impends, interest rates are decreased, the investment rate rises in response and more money enters circulation.

The policy that should be pursued depends critically on what is occurring in the system at the moment. For one to know what is going on requires a feedback of information. The concept of feedback control is now familiar to almost everyone. It means an automatic regulating linkage between some variable and the force producing it. One of the earliest applications of feedback control in technology is the governor used by James Watt on his steam engine. Even earlier Christian Huygens had devised what might be called a static feedback system to regulate the period of a clock pendulum [*see top illustration on next page*].

Usually it is a combination of complexity and ignorance (in polite circles referred to as "uncertainty") that forces one to employ feedback control. If, for example, the workings of the economy were as fully understood, let us say, as the movements of the planets, one could predict well in advance the behavior of producers and consumers; one could predict such things as the effects of population growth and the consequences of introducing new goods and

CHEMICAL PROCESS CONTROL SYSTEM in which a digital computer (*foreground*) exercises closed-loop control is shown in the Bishop, Tex., plant of the Celanese Chemical Company. The plant converts petroleum gases to acetic acid, acetaldehyde and other chemicals that are used in paints, plastics, fibers, drugs, cosmetics, fuels and lubricants. This computer and those at Paradise Station of TVA were built by Bunker-Ramo Corporation. Closed-loop computer systems are also installed in steel-rolling mills.

services. On the basis of this knowledge one could compute and announce the desired interest rate for years ahead. It should be observed that one would then have to reckon with the consequences of publishing the rates in advance, because producers and consumers would include the *future* rates in their *current* economic decisions.

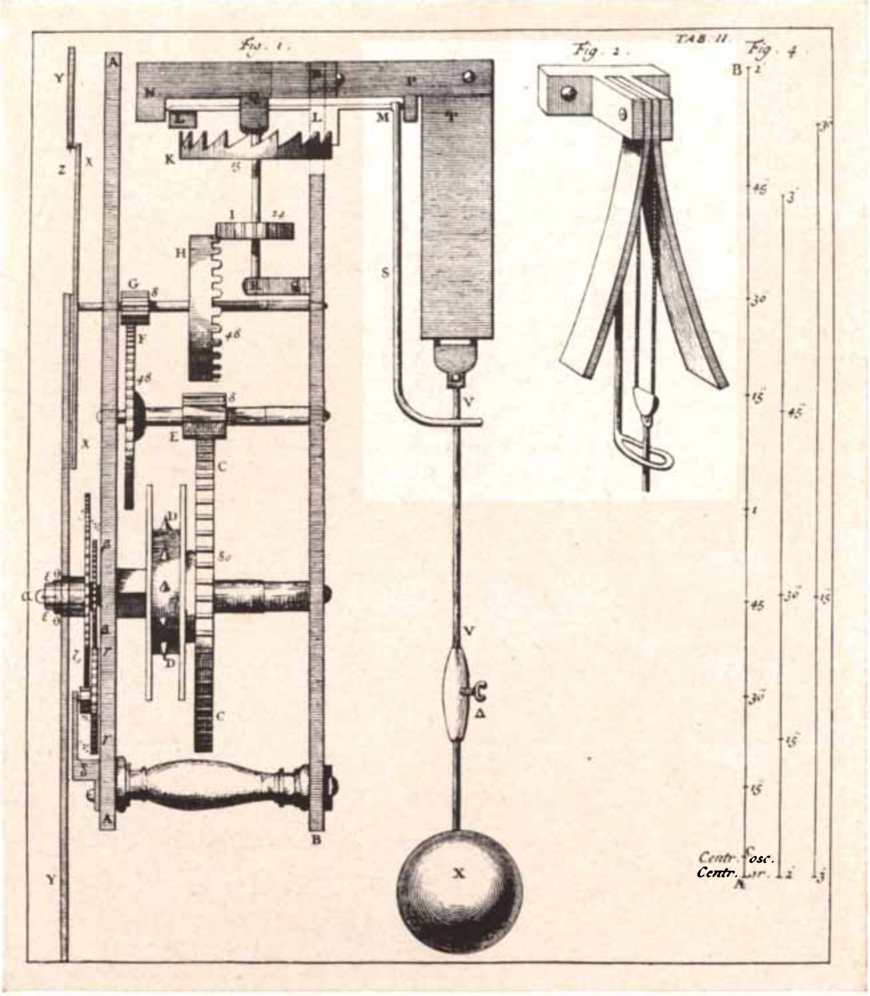
In actuality one must adopt a wait-and-see policy. One observes the economic scene for a period of time and draws conclusions about current trends. On the basis of these conclusions the interest rate, or some other control lever, is changed. One hopes that the action is well timed, or in phase. The matter of the timing of external influences is of crucial importance in control theory. Anyone who has pushed a swing is familiar with the consequences of applying the impulse a fraction of a second too soon or too late.

Since complexity and uncertainty abound in modern control problems, the use of feedback has become routine. In fact, it is sometimes forgotten that control problems are still solved without direct, or active, feedback. This is the case, for example, when an automatic machine tool is set up to turn out a number of identical parts. It is assumed that the control problem is completely solved in advance. In practice, of course, the parts vary slightly and finally exceed the prescribed tolerance, whereupon the machine is readjusted. This is feedback control after the fact.

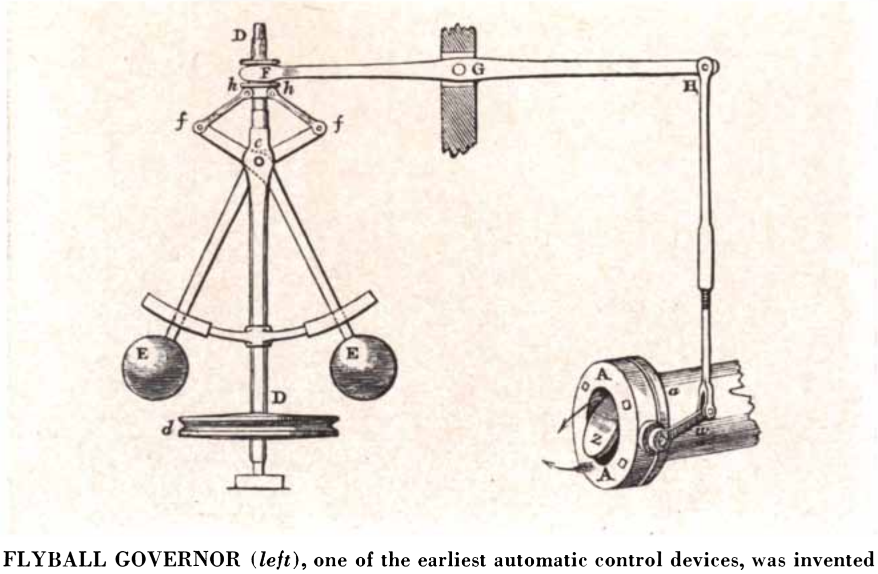
In the newest machine tools the dimensions of the workpiece are monitored continuously, and feedback control is employed to regulate precisely the amount of metal removed. In this way it is possible to turn out parts that are as nearly identical as one might wish. For such jobs digital computers can be used, but they are not essential.

The computer is essential when complex decisions must be made at high speed, as in the launching of a space vehicle. This is a multistage decision process whose solution is contingent on information acquired and fed back to the control system as the process unfolds. A computer, either aboard the rocket or on the ground, is essential for making a succession of decisions as rapidly as may be necessary. Such a computer is said to be operating in "real" time, because it must keep pace with the problem being solved.

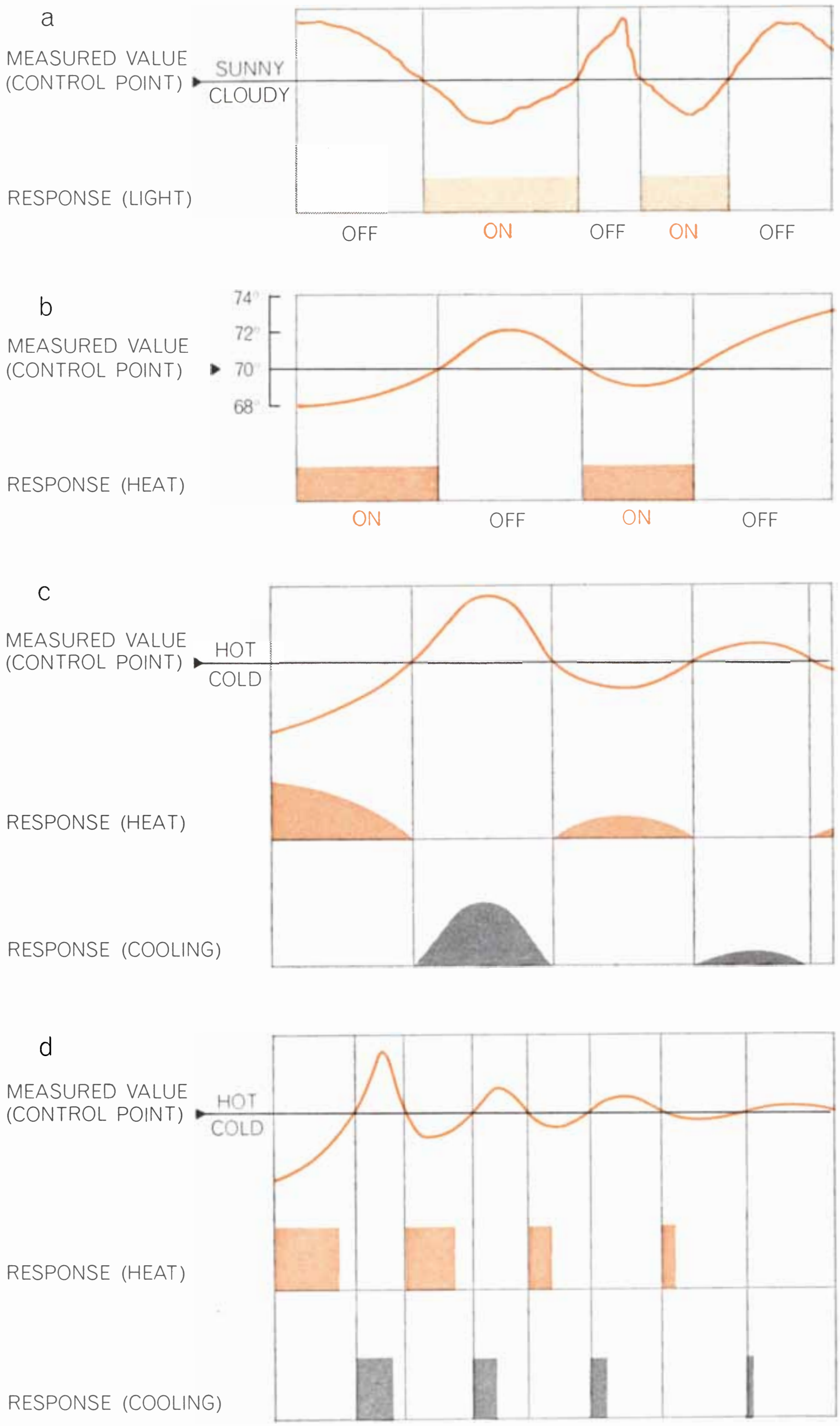
A process-control computer installed in a refinery must also operate in real time, but the time available for making a decision may be 10 or 100 times longer than that available for rocket



FEEDBACK CONTROL for a clock pendulum was invented in 1673 by the Dutch mathematician Christian Huygens. The curved metal strips on each side of the pendulum cords (*seen in perspective at right and labeled "T" in the side elevation of the clockwork at left*) were designed to make the period of the pendulum constant regardless of the length of its arc. The rod *S* moved with the pendulum, transmitting its motion to the clock.



FLYBALL GOVERNOR (*left*), one of the earliest automatic control devices, was invented by James Watt to govern the speed of the steam engine. As the engine speeds up, the rod (*D*) on which the balls (*E*) are mounted spins faster, causing the balls to fly outward. This in turn closes the butterfly valve (*Z*), decreasing the supply of steam to the engine (*not shown*) and slowing it. A fraction of the output of the engine goes into the rotation of the flyballs; a fraction of this fraction is fed back to govern the speed of the engine.



FOUR CONTROL SYSTEMS show how a measured variable can be brought under increasing refinement of control. Diagram *a* depicts a simple on-off response to a measured value, such as turning on the lights in a room when the sun goes behind a cloud. The measured value is not regulated and feedback is not employed. In *b*, which represents a typical home-heating system, on-off response is combined with feedback. When the temperature falls below the desired value, the furnace goes on, but since no cooling system is provided the room temperature may climb higher than desired on a sunny afternoon. The system in *c*, which could represent the heating of a chemical reaction vessel, provides both heating and cooling. The response is graduated so that as the control point is approached the rate of heating or cooling is reduced. The control problem in *d* is the same as in *c* but two modifications have been introduced to improve the speed and accuracy of control. Heating and cooling are not graduated but operate at a constant high rate when called for. This is known as "bang-bang" response. In addition a computer in the control system measures the rate of change in the controlled variable, takes account of the time lag in the temperature-recording mechanism and shuts off the heating or cooling before the control point is reached. Thus oscillation, or "hunting," of the system is damped out quickly.

guidance. On the other hand, a process-control computer may have to deal with 10 or 100 more variables than the rocket computer. And it may have to review a lengthier sequence of logical alternatives before making a decision.

What tools does the mathematician have for trying to control a multistage decision process? The conventional approach can be labeled "enumerative." Each decision can be regarded as a choice among a certain number of variables that determine the state of the process in the next stage; each sequence of choices defines a larger set of variables. By lumping all these choices together the mathematician can "reduce" the problem to a Newtonian one of determining the maximum of a given function.

It would seem simple enough to maximize a reasonably well-behaved function; using the familiar technique of calculus, the mathematician takes partial derivatives and solves the resulting system of equations for the maximizing point. Unfortunately the effective analytic or numerical solution of many equations, even apparently uncomplicated linear ones, is a difficult matter. By itself this is nothing more than the "curse of dimensionality," with which physicists have had to live for many years; significant results can be obtained in spite of it.

There are, however, more serious difficulties. In many cases the solution is a boundary point of the region of variation. This corresponds to the constraints, or restrictions, of real physical and engineering systems. When this is so, calculus is often inadequate for discovering maximum and minimum points and must be supplemented by tedious (and usually impossible) hunt-and-search techniques. Finally there is the frequent complexity that the outcome of a decision is not explicitly determined but is itself a random variable. The process is then said to be stochastic. Here to an even greater extent any simple enumerative technique is doomed by the vast proliferation of possible outcomes at every stage in the process. One cannot enumerate "all" possibilities and choose the best—not when there are 10^{50} or 10^{100} possibilities.

Has the mathematician now reached the end of his resources? Not if he will step back and ask himself if he has understood the nature of the solution he is seeking. How, he must ask himself, is the form of the solution influenced by the physical properties of the system? In other words, the mathematician cannot consider his problem solved until

he has understood the structure of the optimal policy. Let me explain.

We have seen that in the conventional approach the entire multistage decision process is regarded as if it were a one-stage process. Thus if the process has N stages and there are M decisions to be made at each stage, the conventional approach envisages a single-stage process in MN dimensions. What one would like is to avoid this multiplication of dimensions, which stifles analysis, fogs the imagination and inevitably impedes computation.

The alternative approach—the policy approach—places emphasis on the characteristics of the system that determine the decision to be made at *any* stage of the process. In other words, instead of determining the optimal sequence of decisions from some *fixed* state of the system we wish to determine the optimal decision to be made at *any* state of the system. Only if we know the latter state do we understand the intrinsic structure of the solution.

The mathematical virtue of this approach lies first of all in the fact that it reduces the dimension of the problem to its proper level, which is the dimension of the decision that confronts one at any given stage. This makes the problem analytically more tractable and computationally much simpler. In addition this approach provides a type of approximation ("approximation in policy space") that has a unique mathematical property ("monotocity of convergence"). This means that each successive approximation improves performance [*see illustration on page 194*].

The name I proposed some years ago for this policy approach to multistage decision processes is dynamic programming. One of its goals is the determination of optimal feedback control. The adjective "dynamic" indicates that time plays a significant role in the process and that the order of operations may be crucial. The approach is equally applicable, however, to static processes by the simple expedient of reinterpreting them as dynamic processes in which time is artificially introduced.

Dynamic programming has given rise, in turn, to subsidiary and auxiliary control theories that go by a variety of names: theories of stochastic and adaptive variational processes, theories of Markovian decision processes, theories of quasi-linearization and invariant embedding. They cannot be explained in a few words; I mention them merely to indicate how control theory has branched and developed in recent years.

Let me now illustrate how the adoption of a policy can simplify a problem that otherwise would be hard to handle on a computer. (Complex versions of the problem cannot be handled without a computer.) Consider the problem of a hotel manager who wants to provide chairs for a group of people in a room. He has a helper who carries chairs with ease but who cannot count. What does the manager do?

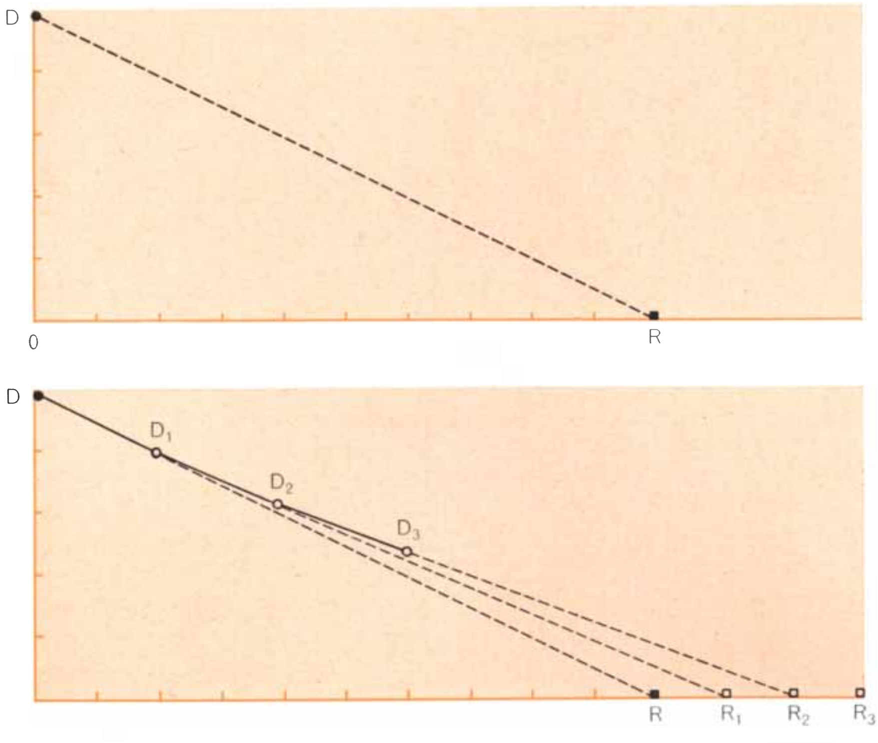
He employs the primitive and powerful concept of equivalence, together with feedback control. He tells the helper to keep bringing chairs until everyone in the room has a chair. This sequential procedure guarantees that each person will have a chair, without ever determining how many people or chairs there are. Furthermore, if some chairs are defective, a simple modification guarantees that everyone will eventually have a sound chair.

Consider next the case of an elderly woman whose memory is failing. It irritates her to have to hunt through her wardrobe for various items of clothing when she dresses in the morning. She could create a filing system, complete with a written index, but this would be a lot of trouble. Instead she solves her problem by putting a complete outfit in every available drawer.

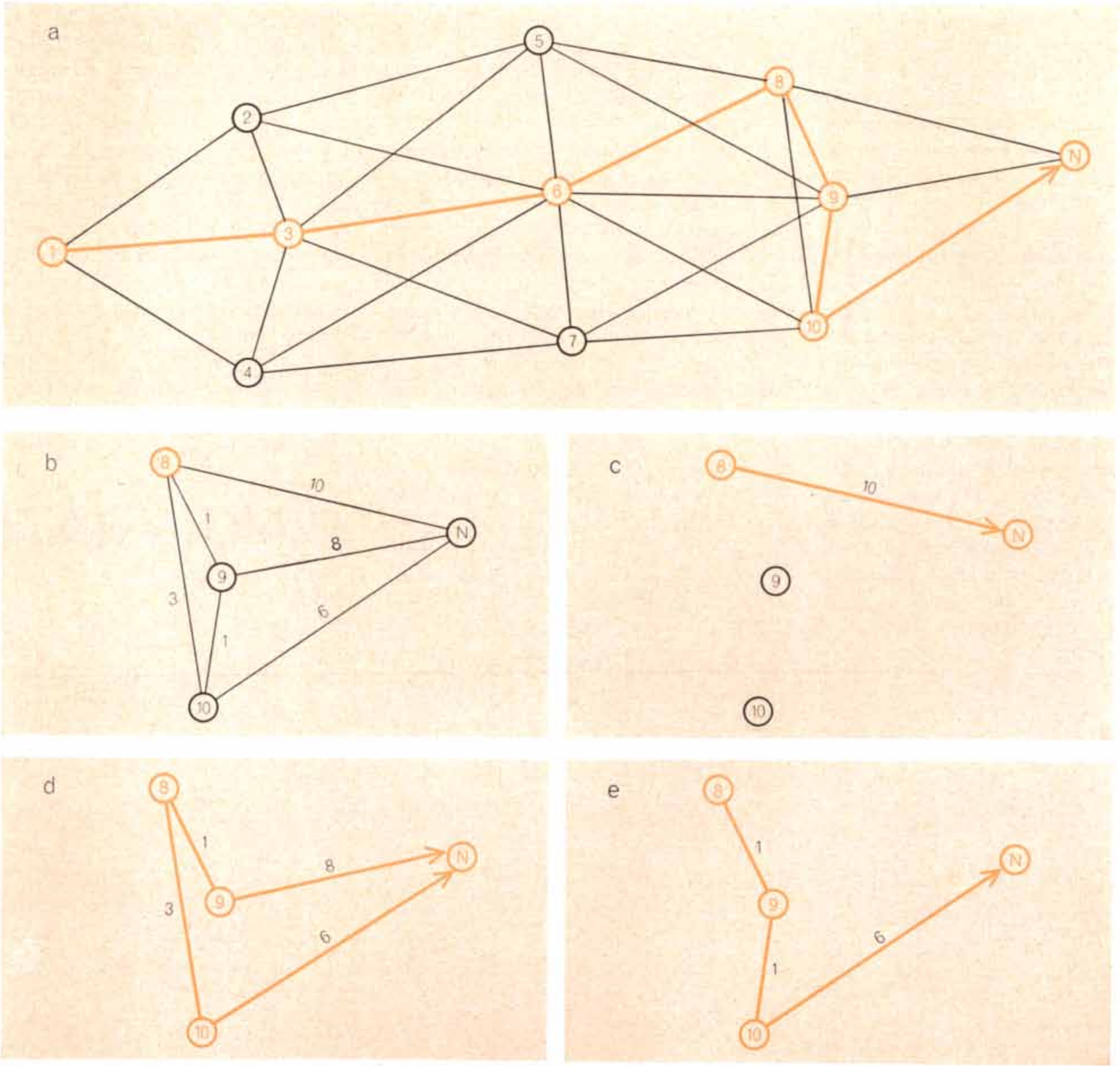
In both cases the solution is quite "simple," but it is not necessarily obvious. Both ideas are currently used in programming computers to solve complex problems. The first is used in certain simulation processes and in Monte Carlo calculations. The second is used for retrieving key items of information from a very large computer memory. Since the items are needed frequently they are stored in several places, thus considerably reducing retrieval time.

I might add that many mathematicians have the nagging suspicion that the universe is much simpler than it appears in their complex mathematical models. It is not easy, however, to capture the fresh view required for the simple approach. In the use of computers, changes in viewpoint such as the two just mentioned have time and again changed an impossible problem into a possible one, a merely difficult one into a routine exercise.

The next example is chosen to show how the concept of policy can not only simplify a multistage decision problem



PURSUIT PROBLEM can be solved by adoption of a simple policy that lends itself to computer implementation. The problem is to find the path traced by a dog (D) chasing a rabbit (R). At the outset (*top*) the rabbit is 100 feet from 0 and the dog is 50 feet from 0. The dog runs at 22 feet per second, the rabbit at 11 feet per second. The dog always continues in a particular direction for one second. After the first second the dog has reached D_1 and the rabbit R_1 (*bottom*). To determine the point D_1 a straight line is drawn between D and R and 22 units are measured along it. Similarly, D_2 is determined by connecting D_1 and R_1, and so on. The resulting path approximates the one taken by the dog and can be refined by changing the direction of the path at shorter and shorter intervals.



ROUTING PROBLEM is commonly met in control theory and was generally difficult to solve before computers and special programming methods were developed. The problem shown here is to find the path from *1* to N that requires the minimum time. The circles represent towns that are connected by a network of roads and the travel time between all pairs of towns is known. The traditional approach to such a problem was simply to enumerate all possibilities. This quickly leads to a "race against N." In the example as drawn, in which N is only 11, the number of different routes from *1* to N (with no backtracking) is more than 10,000. If N were 30, a high-speed computer would need more than 100 hours just to enumerate all the possible routes. One way to make such problems tractable is to use "dynamic programming," which depends on a selection of "policies." The virtue of such policies is that they can be applied from any point i in the network and thus satisfy the injunction: Do the best you can from where you are. The four smaller diagrams show how policies are selected assuming that i is point *8*. The travel times, in hours, for the various routes from *8* to N are shown in *b*. The initial policy (*c*) is to go directly from *8* to N, which takes 10 hours. The second policy (*d*) is to make one stop, which provides two alternative routes of nine hours each. The third policy (*e*) is to make two stops, which provides the minimum-time route. If point *1* were selected as the initial point i, the same procedure would be followed, but since this particular network does not provide a direct path from *1* to N, the first policy that could be examined would be one with the least number of stops, in this case three. There are, of course, ways of formulating this policy approach in terms of a computer program. The equation for solving the problem is

$$f_i = \min_{j \neq i} [t_{ij} + f_j],$$

in which f_i is the minimum time from any point i to N; t_{ij} is the time required to go from i to any other point j, which may be N itself, and f_j is the minimum time to go from j to N. This dynamic programming equation is solved by successive approximations—"approximations in policy space"—in which each successive approximation improves the result. The equation can be solved numerically for networks of several hundred points by hand in a few hours and by electronic computer in a few seconds. The equation determines both the optimal policy ("Where does one go next?") and the minimum time. Moreover, the equation embodies all the mathematical power of the classical calculus of variations.

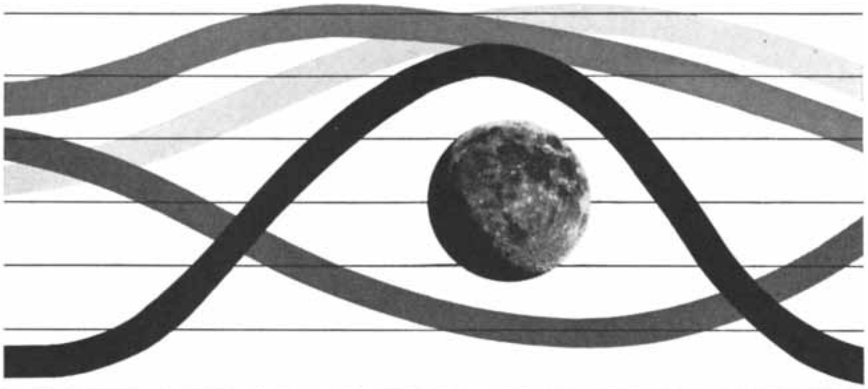
but also yield numerical answers. To follow this example the reader must refer to the illustration on page 192, which shows the path of a dog chasing a rabbit. The dog is initially at D; the rabbit is at R and is running to the right along the x axis. If the dog always heads straight toward the rabbit, what curve does the dog follow?

This is a standard problem in the theory of differential equations, but the nonmathematical reader would hardly be edified if he were given the explicit form of the solution. To understand it requires a certain amount of mathematical training. This is strange when one thinks about it; the dog solves the problem without hesitation, although of course he does not get numerical answers.

It is easy to obtain a good approximation of the shape of the curve in the following way. Let us assume that the dog can run 15 miles per hour, or 22 feet per second, and that the rabbit can run just half as fast, or 11 feet per second. The rabbit is originally 100 feet from 0 and the dog is 50 feet from 0 at a point perpendicular to the x axis. Assume now that the dog continues in any particular direction for one second at a time. At the end of one second the dog has reached D_1 and the rabbit R_1. Another second later the dog has reached D_2, the rabbit R_2 and so on.

The point D_1 is determined by connecting D and R with a ruler and measuring 22 units along it. Similarly, D_2 is determined by connecting D_1 and R_1 and repeating the same measurement. The process is continued until the distance between dog and rabbit is closed. (We will ignore the fact that the closing stages are made a little messy by this method.) The broken-line path is a simple approximation of the actual path traversed by the dog. It is evident that the approximation can be made as close as desired by carrying out the change in the dog's direction at shorter and shorter time intervals, say every hundredth or every millionth of a second. By hand computation this would be increasingly tedious, but an electronic computer can do the job easily in a matter of seconds.

More sophisticated versions of this problem occur in the determination of optimal trajectories for space vehicles. In some of these cases the "rabbit" is imaginary and the problem is to determine where to point to achieve a desired course; in other cases the "rabbit" is real enough—another craft or a planet, perhaps—and establishing exactly where it



is provides a further significant complication.

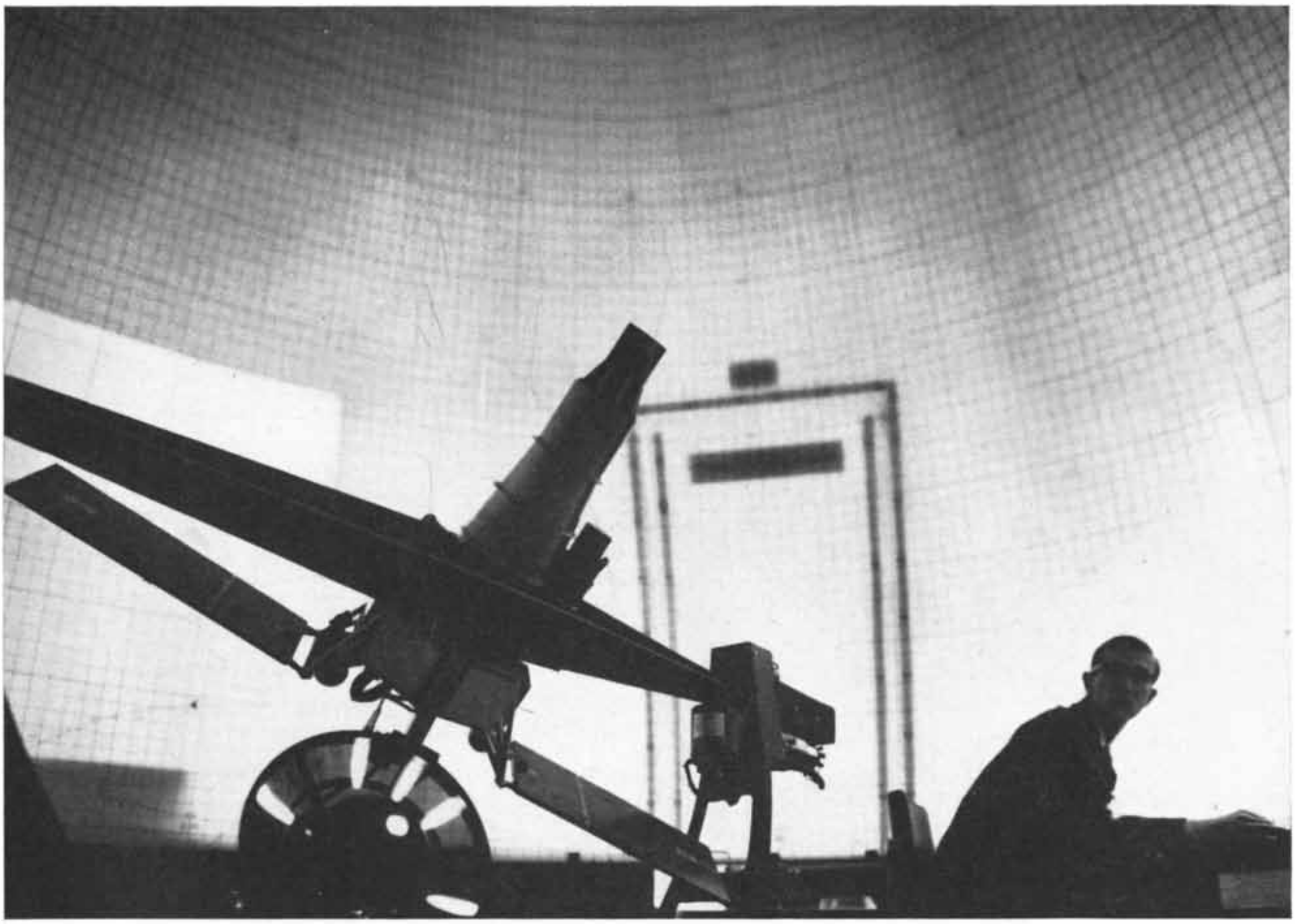
The point I wish to emphasize is that one can obtain a solution to the original problem by concentrating on the original process. One merely follows the instructions for what to do at every point in time and space. In mathematical terms, one carries out a policy.

The importance of this from the standpoint of control theory is manifold. In the first place, it is easy to use computers to implement policies. In the second place, the mathematical level is more fundamental, deeper and yet relatively uncomplicated by symbol manipulation. Policies are invariably simpler than time histories. Much more emphasis is now placed on the formulation of the problem. The idea is to take full advantage of the structure of the process in order to describe it in a most convenient analytical fashion and in order to make clear the structure of the optimal policy. One tries to avoid any routine description in terms of complicated equations that do not easily yield to numerical approach. One does not try to fit every new type of decision process into the rigid mold of 18th-century mathematics. This is the policy concept behind dynamic programming.

With this concept, which recognizes the resources of the digital computer and accepts it as an ally, one can easily and quickly obtain the numerical solution of control problems in many different fields that defied even the most resourceful mathematicians 20 years ago. The new approach has made it possible to solve formidable problems in trajectory analysis, process control, equipment replacement and inspection procedures, communication theory, the allocation of water resources and hydroelectric power, the use of forest resources and investment planning—to mention only a few important areas.

Beyond this, the concept of policy can readily be applied to study the more difficult and realistic classes of decision processes involving uncertainty and learning. I have already referred to the former as stochastic processes; the latter are known as adaptive control processes.

The dog-rabbit pursuit process was an example of a deterministic process in which the basic mechanisms are fully understood and it is "merely" a matter of devising a suitable procedure for solving it. Thus we assumed that the positions of both the dog and the rabbit can be precisely observed and determined at each instant, that the



speeds remain constant, that nothing distracts the animals and so forth.

Even this idealized situation leads to difficult enough mathematics and a plethora of unsolved problems, as the study of classical celestial mechanics has demonstrated. After centuries of observation and computing, the positions of the planets next year or 10 years from now cannot be predicted with the accuracy desired. There will be a significant discrepancy between their predicted positions and those actually observed. A more upsetting problem for long-term planners is that no one knows if the solar system is completely stable.

It is obvious that if the idealized situation of perfect information and prediction cannot be found in the planetary motions, it can hardly be found in problems of trajectory optimization, satellite control and space rendezvous, much less in chemical process control, economic planning and medical diagnosis. In practice we are constantly using fallible devices for sensing and measuring, for processing, storing and retrieving information and for carrying out control decisions. Thus at every step we introduce error: error in observation, in calculation, in decision, in operation and even in the evaluation of outcomes.

The concept of a policy involving feedback control is ideally designed to handle the certain uncertainties of the actual world. By means of dynamic programming the injunction "Do the best you can in terms of where you are" (which is eminently sound common sense) can be readily translated into algorithms, or sets of rules, for the rigorous formulation and numerical solution of stochastic control problems.

When we turn to adaptive control processes we find a still higher order of uncertainty. In the stochastic case it is tacitly assumed that we know the detailed structure of the system we are studying, that we know various causes and various effects and, perhaps most essential of all, that we know what we want to do. In the case of an adaptive control process none of these assumptions may be valid.

Virtually all the unsolved major health problems can be regarded as adaptive control problems. Since no one knows the causes of cancer, coronary disease or mental illness, therapies aimed at control are necessarily based on a wide variety of hypotheses. This explains, of course, why so much caution must be exercised in treating patients. In the study of our national economy no one knows exactly what

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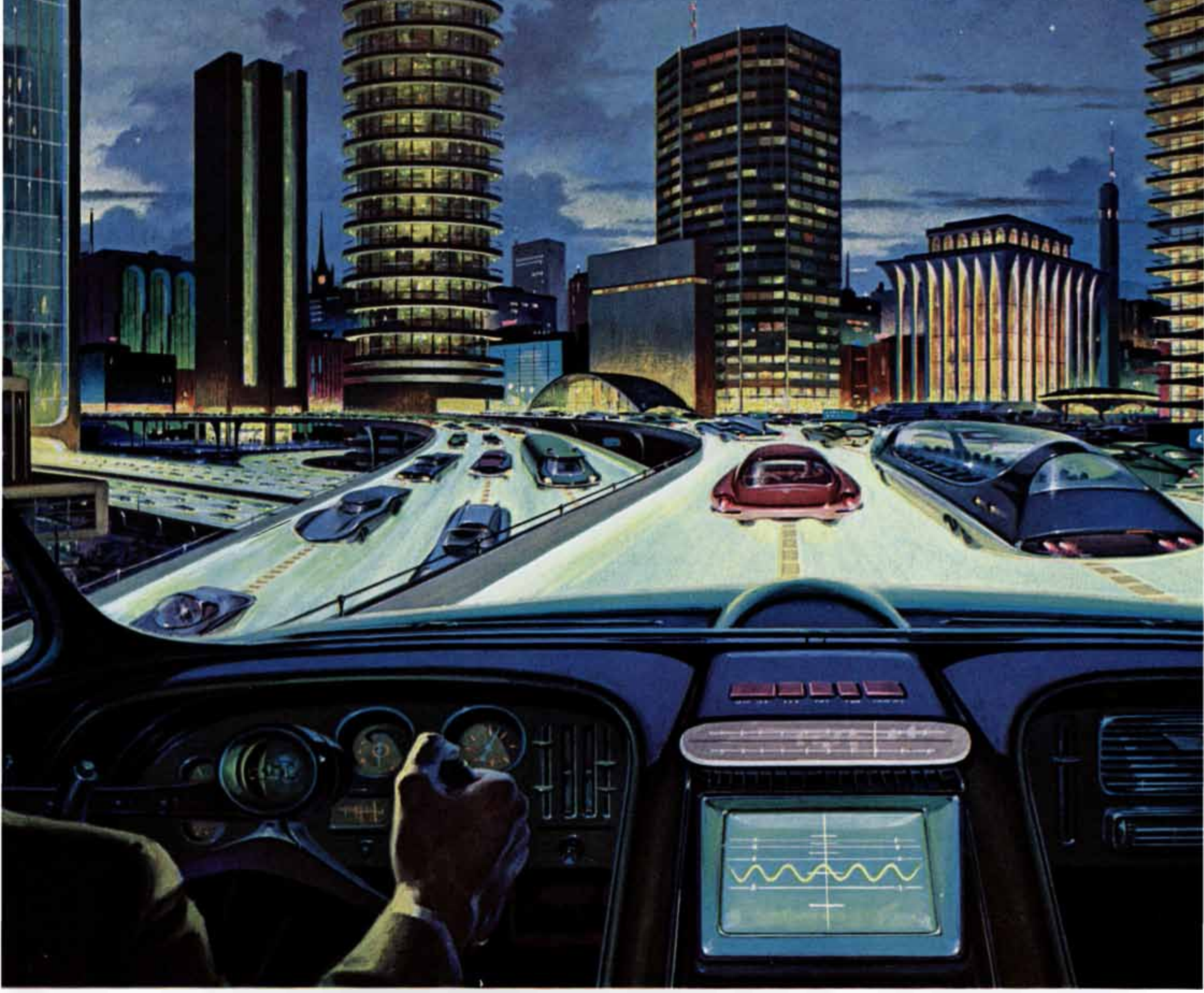
will happen if taxes are cut or if military spending is reduced. Furthermore, there is a continuing controversy even about what constitutes a desirable economic condition.

When faced with an adaptive control problem, one expects to learn more about the system as time goes on and to modify one's policies accordingly. All major decision processes in life are adaptive control processes. It should not be surprising, therefore, that biological evolution has equipped animals to deal more or less successfully with adaptive control problems. Deterministic and, to a degree, stochastic control problems can be handled by animals on the basis of instinct. Instinct can be described as feedback control of a deterministic type. The same stimulus produces the same reaction, regardless of what else has changed in the environment.

To handle adaptive control problems the higher animals are equipped with something we identify as "intelligence." In fact, intelligence can be defined as the capacity to solve, in some degree, an adaptive control problem. Intelligence manifests itself by adaptation, by flexible policies. It is difficult, of course, to draw a sharp dividing line between instinct and intelligence. It is probably better, then, to call every type of feedback behavior "intelligence" and subsequently distinguish between levels of intelligence.

Norbert Wiener, the eminent mathematician who died last winter, formulated the provocative idea that it should be possible to develop a unified theory of feedback control applicable both to living organisms and to machines. To express this idea he coined the term "cybernetics." It was his hope, shared by others, that techniques used so successfully in control engineering could be applied to biomedical problems (for example the design of artificial human organs) and also that research into neurophysiology might provide valuable clues in the design and study of communication systems, computers and more general control systems of all kinds. But as mathematicians, physiologists and engineers explore the subtle difficulties of dealing with large-scale systems—living and nonliving—of different degrees of complexity, it seems less and less likely that any single "cybernetic" theory will serve all purposes. Nonetheless, for those who want to understand both modern science and modern society, there is no better place to start than control theory.



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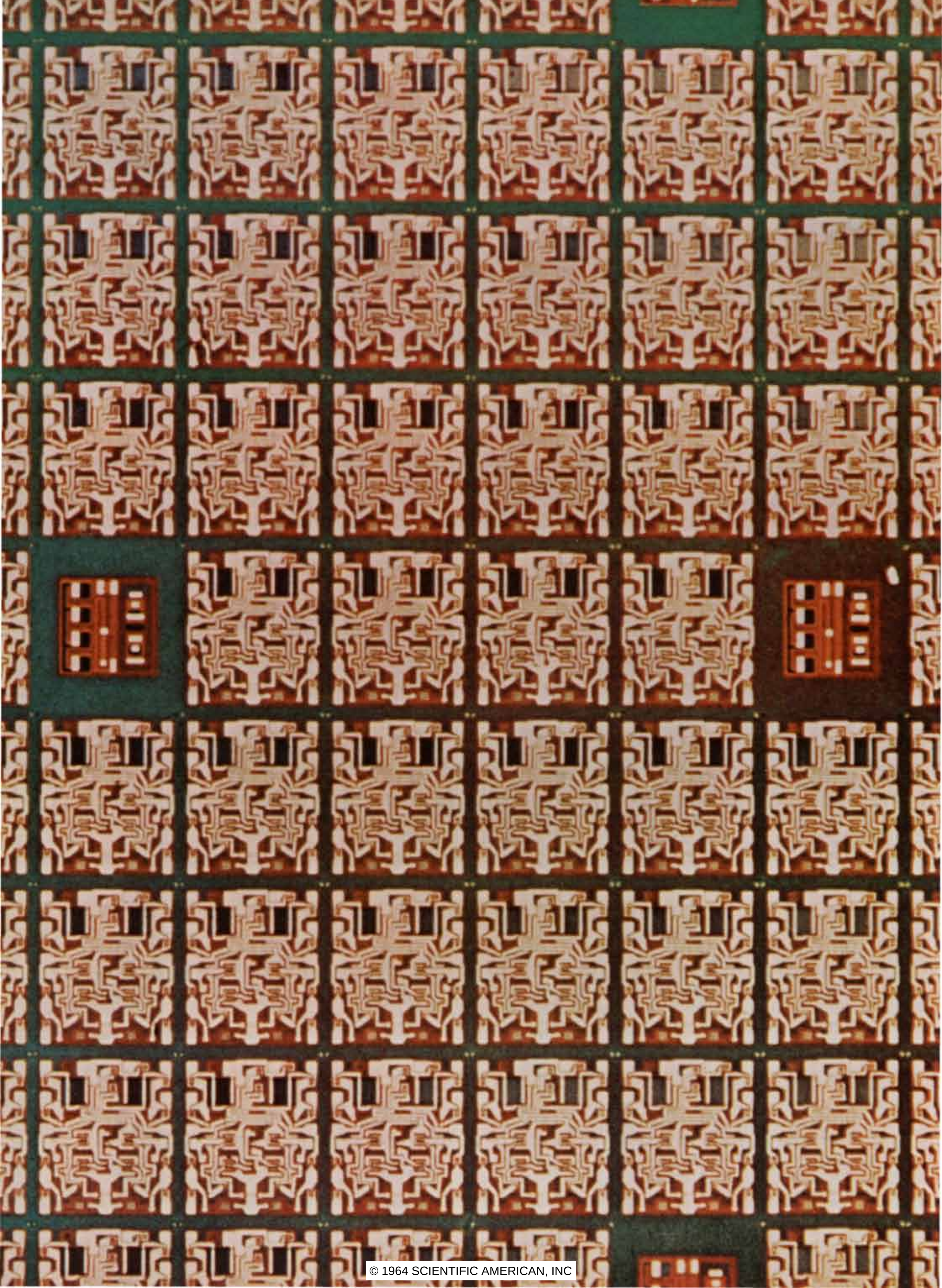


COMPUTERS

These machines that do arithmetic at high speed have evolved in response to the need to apply mathematics. Now they also promise to play a role in the progress of mathematics itself

by Stanislaw M. Ulam

Although to many people the electronic computer has come to symbolize the importance of mathematics in the modern world, few professional mathematicians are closely acquainted with the machine. Some, in fact, seem even to fear that individual scientific efforts will be pushed into the background or replaced by less imaginative, purely mechanical habits of research. I believe such fears to be quite groundless. It is preferable to regard the computer as a handy device for manipulating and displaying symbols. Even the most abstract thinkers agree that the simple act of writing down a few symbols on a piece of paper facilitates concentration. In this respect alone—and it is not a trivial one—the new electronic machines enlarge our effective memory and provide a marvelous extension of the means for experimenting with symbols in science. In this article I shall try to indicate how the computer can be useful in mathematical research.

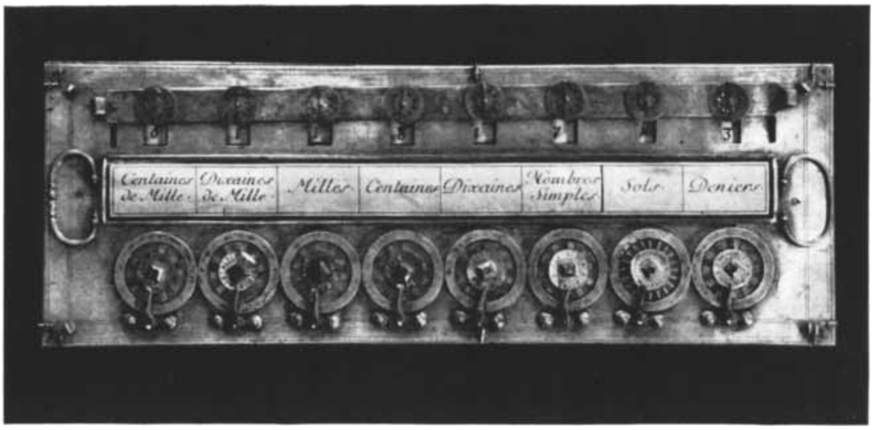
The idea of using mechanical or semiautomatic means to perform arithmetical calculations is very old. The origin of the abacus is lost in antiquity, and computers of some kind were evidently built by the ancient Greeks. Blaise Pascal in the 17th century constructed a working mechanism to perform arithmetical operations. Gottfried Wilhelm von Leibniz, one of the creators of mathematical logic as well as the coinventor of the infinitesimal calculus, outlined a program for what would now be called automatized thinking. The man who clearly visualized a general-purpose computer, complete with a flexible programming scheme and memory units, was Charles Babbage of England. He described a machine he called the analytical engine in 1833 and spent the rest of his life and much of his fortune trying to build it.

Among the leading contributors to modern computer technology were an electrical engineer, J. Presper Eckert, Jr., a physicist, John W. Mauchly, and one of the leading mathematicians of this century, John von Neumann. In 1944 Eckert and Mauchly were deep in the development of a machine known as ENIAC, which stands for Electronic Numerical Integrator and Computer. Designed to compute artillery firing tables for the Army Ordnance Department, ENIAC was finally completed late in 1945. It was wired to perform a specific sequence of calculations; if a different sequence was needed, it had to be extensively rewired. On hearing of the ENIAC project during a visit to the Aberdeen Proving Ground in the summer of 1944, von Neumann became fascinated by the idea and began developing the logical design of a computer capable of using a flexible stored program: a program that could be changed at will without revising the computer's circuits.

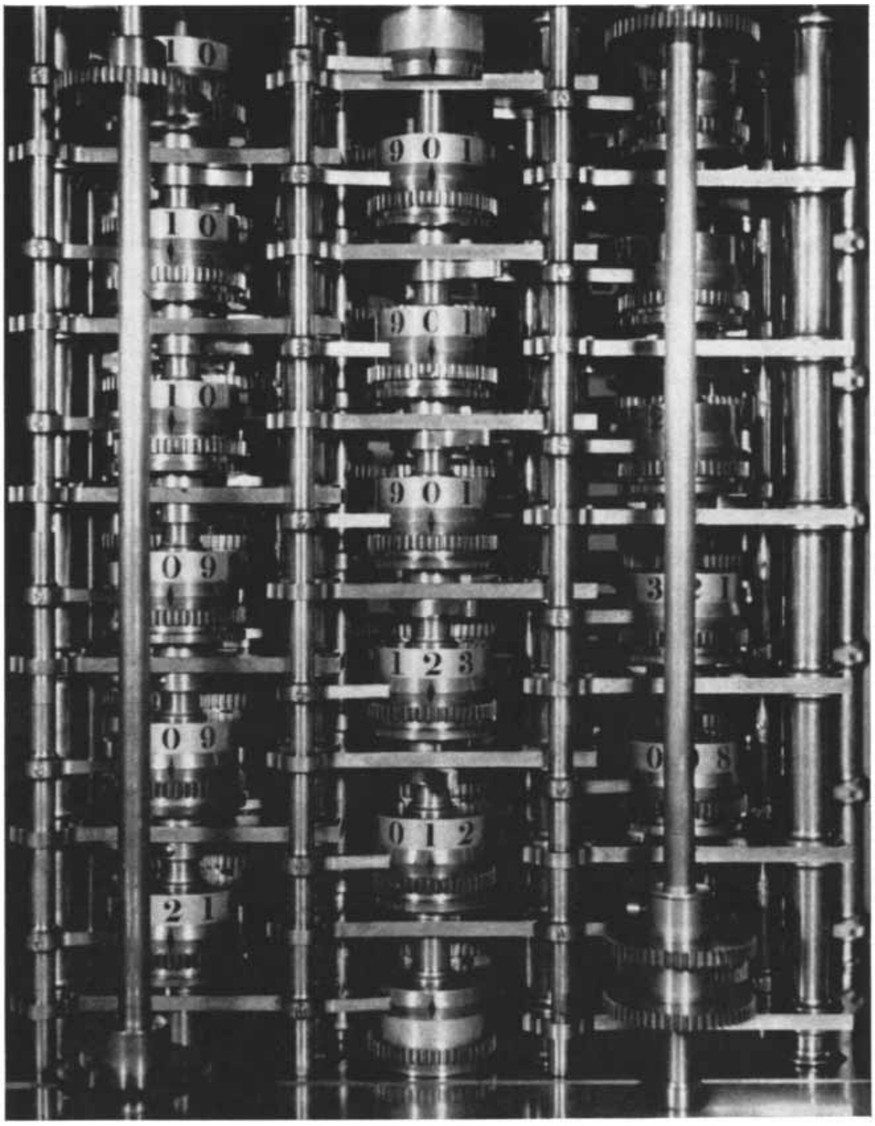
A major stimulus for von Neumann's enthusiasm was the task he faced as consultant to the theoretical group at Los Alamos, which was charged with solving computational problems connected with the atomic-bomb project. After a discussion in which we reviewed one of these problems von Neumann turned to me and said: "Probably in its solution we shall have to perform more elementary arithmetical steps than the total in all the computations performed by the human race heretofore." I reminded him that there were millions of schoolchildren in the world and that the total number of additions, multiplications and divisions they were obliged to perform every day over a period of a few years would certainly exceed that needed in our problem. Unfortunately we could not harness this great reservoir of talent for our purposes, nor could we in 1944 command the services of an electronic computer. The atomic-bomb calculations had to be simplified to the point where they could be solved with paper and pencil and the help of old-fashioned desk calculators.

Down the hall from my present office at the Los Alamos Scientific Laboratory is an electronic computer known as MANIAC II (Mathematical Analyzer, Numerical Integrator and Computer), an advanced version of MANIAC I, which von Neumann and his associates completed at the Institute for Advanced Study in 1952. MANIAC II, which was put in operation in 1957, can add two numbers consisting of 13 decimal digits (43 binary digits) in about six microseconds (six millionths of a second). In a separate building nearby is a still newer computer called STRETCH, built by the International Business Machines Corporation, which can manipulate numbers containing 48 binary digits with about 10 times the overall speed of MANIAC II.

COMPUTER CIRCUITS have become almost microscopic. Although each of the Westinghouse binary integrated circuits (*square units*) on the opposite page is smaller than the head of a pin, it contains six transistors, 12 diodes, 11 resistors and two capacitors. The tiny units are now employed primarily in special computers for military and space applications. More than 100 of them are made at one time on a thin silicon wafer the size of a half-dollar. The four patterns that do not match the others are used for alignment and testing during fabrication. In commercial computers such circuits are expected to provide greater speed and reliability at lower cost than the larger circuits in common use today.



FIRST MECHANICAL COMPUTER was probably this adding machine, designed in 1642 by the French philosopher and mathematician Blaise Pascal. The machine adds when the wheels are turned with a stylus. Gears inside automatically "carry" numbers from one wheel to the next. Similar but somewhat simpler devices, made of plastic, are widely sold.



"DIFFERENCE ENGINE," often called the first modern mathematical machine, was conceived in 1820 by the English mathematician Charles Babbage. He built a small version of it but the larger engine he envisioned was never completed. Parts of it, such as this unit, are now in South Kensington Science Museum. Babbage spent many years trying unsuccessfully to create an "analytical engine" that would do almost everything the modern computer does.

MANIAC II and STRETCH are examples of dozens of custom-designed computers built throughout the world in the past 20 years. The first of the big commercially built computers, UNIVAC I, was delivered to the Bureau of the Census in 1951; three years later the General Electric Company became the first industrial user of a UNIVAC I. In the 13 years since the first UNIVAC more than 16,000 computer systems of various makes and sizes have been put to work by the U.S. Government, industry and universities. Of these about 250 are of the largest type, with speed and power roughly comparable to MANIAC II.

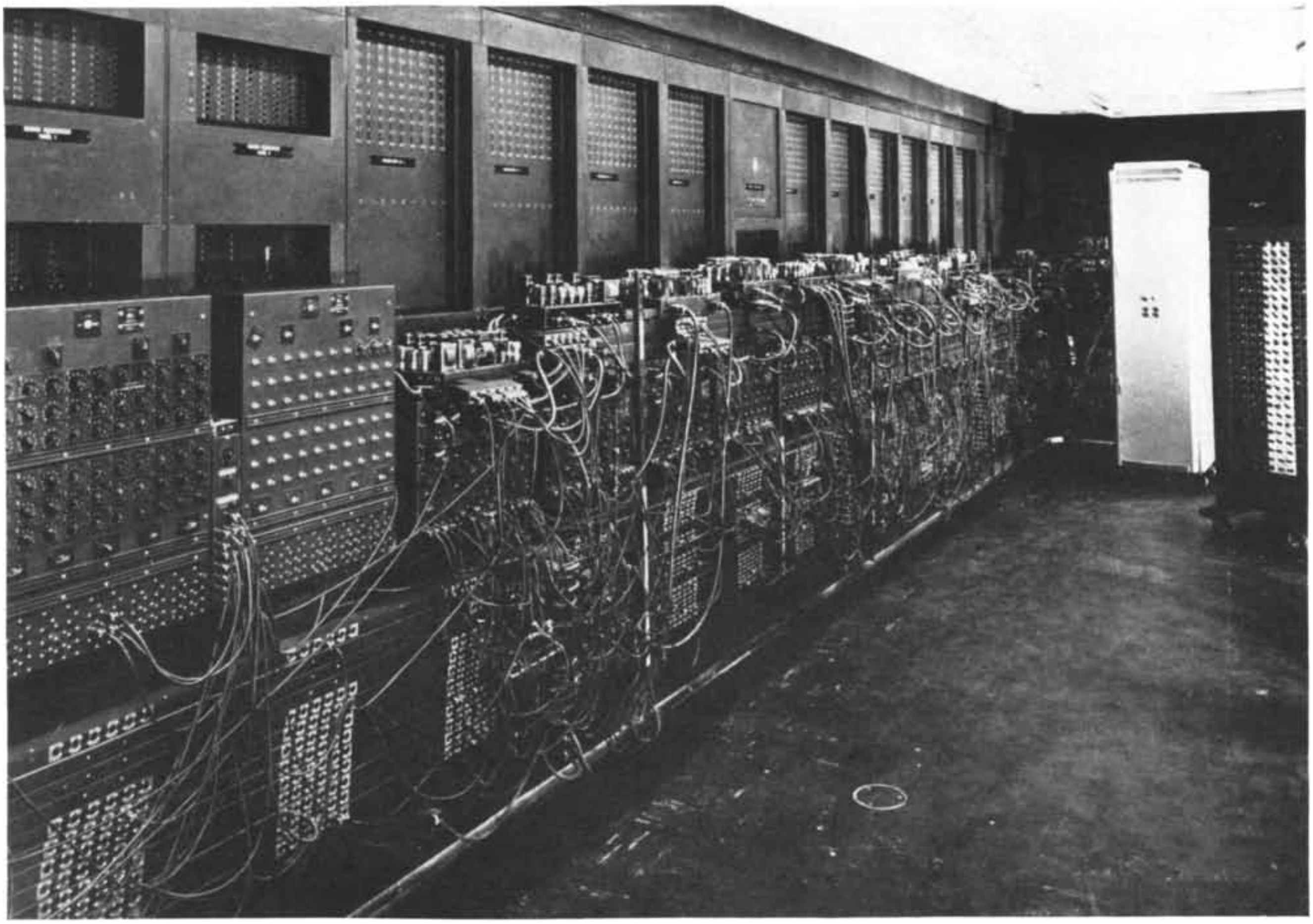
Together with increases in arithmetical speed have come increases in memory capacity and in speed of access to stored numbers and instructions. In the biggest electronic machines the memory capacity is now up to about 100,000 "words," or several million individual binary digits. I am referring here to the "fast" memory, to which the access time can be as short as a microsecond. This time is steadily being reduced; a hundredfold increase in speed seems possible in the near future. A "slow" memory, used as an adjunct to the fast one, normally consists of digits stored on magnetic tape and can be of almost unlimited capacity. The size of memory devices and basic electronic circuits has been steadily reduced, until now even the most elaborate computers can fit into a small room. The next generation of computers, employing microelectronic circuits, will be smaller by a factor of 100 to 1,000.

It is apparent that many problems are so difficult that they would tax the capacity of any machine one can imagine being built in the next decade. For example, the hydrodynamics of compressible fluids can be studied reasonably well on existing machines if the investigation is limited to problems in two dimensions, but it cannot be studied very satisfactorily in three dimensions. In a two-dimensional study one can imagine that the fluid is confined in a "box" that has been divided into, say, 10,000 cells; the cells are expressed in terms of two coordinates, each of which is divided into 100 parts. In each cell are stored several values, such as those for density and velocity, and a new set of values must be computed for each successive chosen unit of time. It is obvious that if this same problem is simply extended to include a third dimension, storage must be provided for a million cells, which exceeds the capacity of present machines. One of the studies that is limited in this way is the effort to forecast the weather, for which it would be desirable to use a many-celled three-dimensional model of the atmosphere.

Sometimes when a problem is too complex to be solved in full detail by computer, it is possible to obtain a representative collection of specific solutions by the "Monte Carlo" method. Many years ago I happened to consider ways of calculating what fraction of all games of solitaire could be completed satisfactorily to the last card. When I could not devise a general solution, it occurred to me that the problem could be examined heuristically, that is, in such a way that the examination would at least give an idea of the solution. This would involve actually playing out a number of games, say 100 or 200, and simply recording the results. It was an ideal task for a computer and was at the origin of the Monte Carlo method.

This method is commonly applied to problems of mathematical physics such as those presented by the design of nuclear reactors. In a reactor neutrons are released; they collide, scatter, multiply and are absorbed or escape with various probabilities, depending on the geometry and the composition of the fuel elements and other components. In a complicated geometry no way is known to compute directly the number of neutrons in any given range of energy, direction and velocity. Instead one resorts to a sampling procedure in which the computer traces out a large number of possible histories of individual particles. The computer does not consider all the possible things that might happen to the particle, which would form a very complicated tree of branching eventualities, but selects at each branching point just one of the eventualities with a suitable probability (which is known to the physicist) and examines a large class of such possible chains of events. By gathering statistics on many such chains one can get an idea of the behavior of the system. The class of chains may have to be quite large but it is small compared with the much larger class of all possible branchings. Such sampling procedures, which would be impracticable without the computer, have been applied to many diverse problems.

The variety of work in mathematical physics that has been made possible in recent years through the use of computers is impressive indeed. Astronomy journals, for instance, contain an increasing number of computer results bearing on such matters as the history of stars, the motions of stars in clusters, the complex behavior of stellar atmospheres and the testing of cosmological theories. It has long been recognized that it is mathematically difficult to obtain particular solutions to problems involving the general theory of relativity so that the predictions of alternative formulations can be tested by observation or experiment. The computer is



FIRST ELECTRONIC DIGITAL COMPUTER, the Electronic Numerical Integrator and Computer (ENIAC), was built at the University of Pennsylvania for the Army Ordnance Department. Completed in the fall of 1945, it had 19,000 vacuum tubes, 1,500 relays and hundreds of thousands of resistors, capacitors and inductors. It consumed almost 200 kilowatts of electric power. Power and tube failures and other difficulties plagued its first few years of operation. To change its program it was necessary to rewire thousands of circuits. With constant improvements ENIAC was kept in service at the Ballistic Research Center, Aberdeen, Md., until late 1955.

now making it possible to obtain such predictions in many cases. A similar situation exists in nuclear physics with regard to alternative field theories.

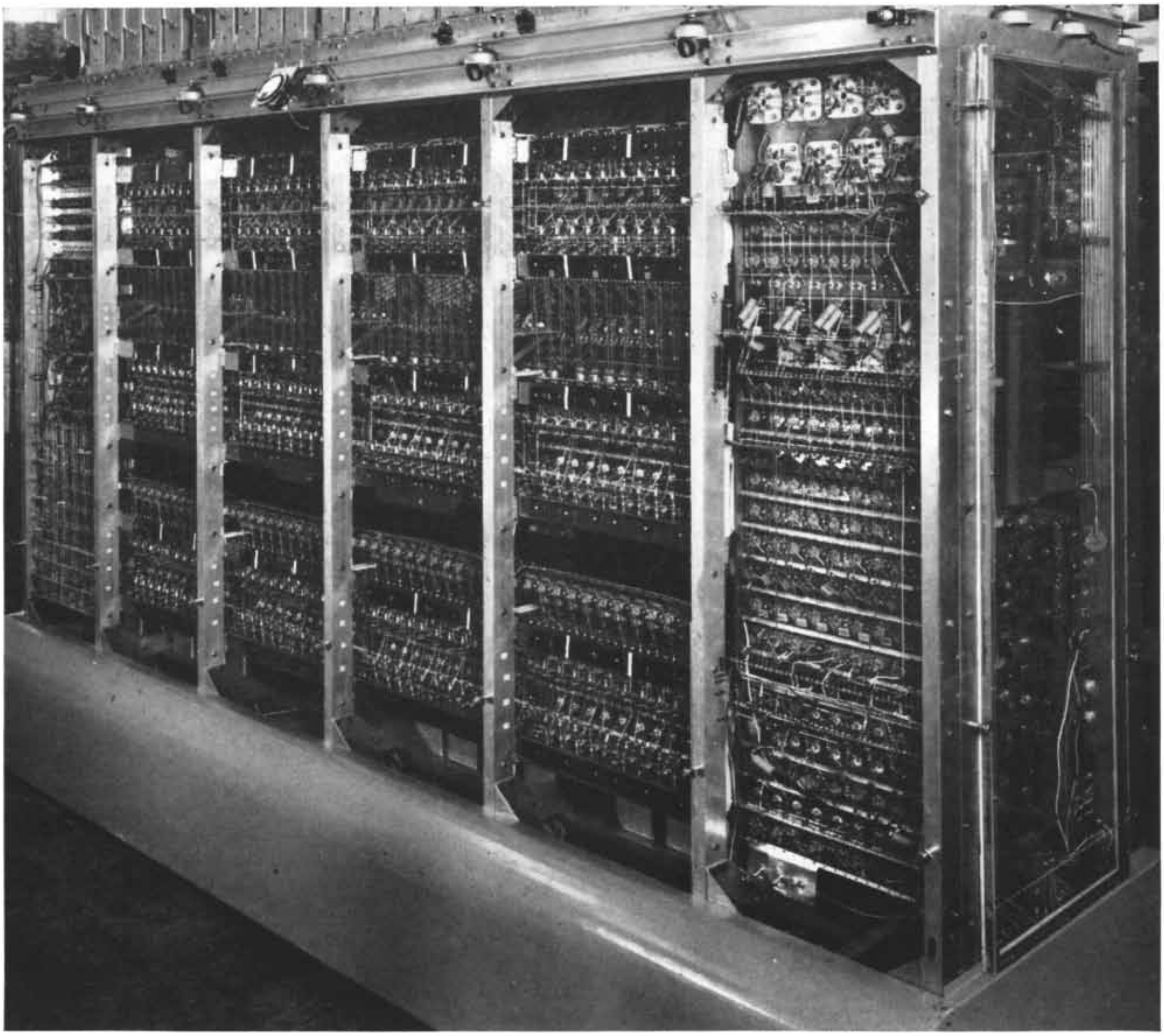
I should now like to discuss some particular examples of how the computer can perform work that is both interesting and useful to a mathematician. The first examples are problems in number theory. This subject deals with properties of ordinary integers and particularly with those properties that concern the two most fundamental operations on them: addition and multiplication.

As in so much of "pure" mathematics the objective is to discover and then prove a theorem containing some general truth about numbers. It is often easy to see a relation that holds true in special cases; the task is to show that it holds true in general.

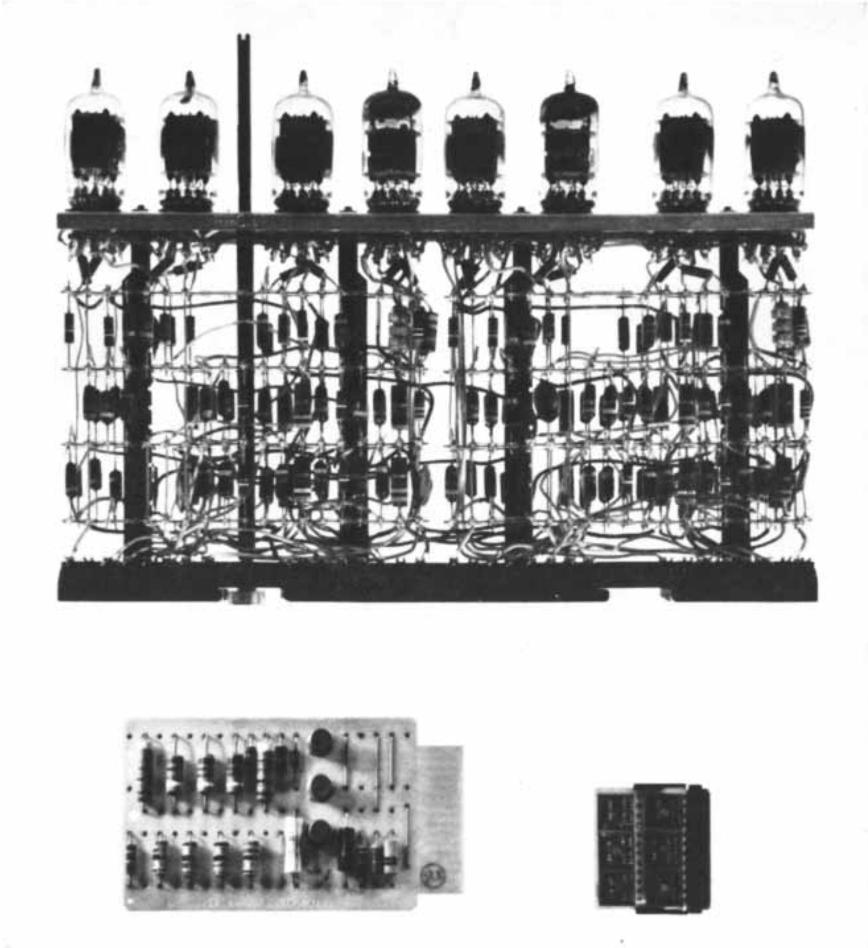
Karl Friedrich Gauss, called "the prince of mathematicians" by his contemporaries, greatly favored experiments on special cases and diligent work with examples to obtain his inspirations for finding general truths in number theory. Asked how he divined some of the remarkable regularities of numbers, he replied, *"Durch planmässiges tattonieren"* —through systematic trying. Srinivasa Ramanujan, the phenomenal Indian number theorist, was equally addicted to experimentation with examples. One can imagine that in the hands of such men the computer would have stimulated many more discoveries in number theory.

A fascinating area of number theory is that dealing with primes, the class of integers that are divisible only by themselves and by one. The Greeks proved that the number of primes is infinite, but even after centuries of work some of the most elementary questions about primes remain unanswered.

For example, can every even number be represented as the sum of two primes? This is the famous Goldbach conjecture. Thus $100 = 93 + 7$ and $200 = 103 + 97$. It has been shown that all even numbers smaller than 2,000,000 can be represented as the sum of two primes, but there is no proof that this holds true for *all* even integers.



MANIAC II (Mathematical Analyzer, Numerical Integrator and Computer) was built at the Los Alamos Scientific Laboratory in 1957. STRETCH, built by the International Business Machines Corporation and installed at Los Alamos four years later, is about 10 times faster than MANIAC II. Both have been used extensively by the author and his colleagues for experimentation in mathematics.



SHRINKAGE OF COMPONENTS has meant greater reliability and speed plus substantial savings in construction and operation of computer systems. The vacuum-tube assembly at top was used in first generation of computers built by International Business Machines Corporation, starting in 1946. First transistorized computers, built in 1955, used circuits such as that at lower left. At lower right is a card of six microminiaturized circuits, each containing several transistors and diodes, which is going into the newest IBM computers.

It is an interesting fact that there are many pairs of primes differing by two, for instance 11 and 13, 17 and 19, 311 and 313. Although it might seem simple to show that there are infinitely many such pairs of "twin primes," no one has been able to do it. These two unsolved problems demonstrate that the inquiring human mind can almost immediately find mathematical statements of great simplicity whose truth or falsehood are inordinately difficult to decide. Such statements present a continual challenge to mathematicians.

The existence of a proof does not always appease the mathematician. Although it is easily proved that there is an infinite number of primes, one would like to have a formula for writing down an arbitrarily large prime. No such formula has been found. No mathematician can now write on demand a prime with, say, 10 million digits, although one surely exists.

One of the largest known primes was found not long ago with the help of an electronic computer in Sweden. It is $2^{3217} - 1$, a number containing 967 digits. A number of this form, $2^n - 1$, is called a Mersenne number. There may be an infinite number of primes of this form. No one knows.

Other special numbers that may or may not yield many primes are Fermat numbers, which have the form $2^{2^n} + 1$. For n's of 0, 1, 2 and 3 the corresponding Fermat numbers are 3, 5, 17 and 257. Even for moderate values of n Fermat numbers become extremely large. It is not known, for instance, if the Fermat number with an n of 13 is a prime (the number is $2^{2^{13}} + 1$, or $2^{8192} + 1$).

It is convenient for computer experimentation that both Mersenne and Fermat numbers have a particularly simple appearance when they are written in binary notation [*see top illustration on next page*]. Fermat numbers start with a 1, are followed by 0's and end with a 1. Mersenne numbers in binary notation consist exclusively of 1's. With computers it is an easy matter to study empirically the appearance of primes written in binary form.

The following statement is most likely true: There exists a number n such that an infinite number of primes can be written in a binary sequence that contains exactly n 1's. (The number of 0's interspersed among the 1's, of course, would be unlimited.) Although this statement cannot be proved with the present means of number theory, I suspect that experimental work with a computer might provide some insight into the behavior of binary sequences containing various numbers of 1's. The following experience may help to explain this feeling.

A few years ago my colleague Mark B. Wells and I planned a computer program to study some combinatorial properties of the distribution of 0's and 1's in prime numbers when expressed in binary form. One day Wells remarked: "Of course, one cannot expect the primes to have, asymptotically, the same number of ones and zeros in their development, since the numbers divisible by three have an even number of ones." This statement was based on the following argument: One would expect a priori that in a large sample of integers expressed in binary form the number of 1's and 0's ought to be randomly distributed and that this should also be the case for a large sample of primes. On the other hand, if it were true that all numbers divisible by three contain an even number of 1's, then the distribution of 1's and 0's in a large sample of primes should not be random.

Returning to my office, I tried to prove Wells's statement about numbers divisible by three but was unsuccessful. After a while I noticed that the statement is not even true. The first number to disprove it is 21, which has three 1's in its binary representation [*see middle illustration on next page*].

Nevertheless, a great majority of the integers divisible by three seem to have an even number of 1's. Beginning with this observation, Wells managed to prove a general theorem: Among all the integers divisible by three from 1 to 2^n, those that have an even number of 1's always predominate, and the difference between their number and the number of those with an odd number of 1's can be computed exactly: it is

MERSENNE NUMBER ($2^n - 1$)			FERMAT NUMBER ($2^{2^n} + 1$)		
n	DECIMAL	BINARY	n	DECIMAL	BINARY
1	1	1	0	3	11
2	3	11	1	5	101
3	7	111	2	17	10001
4	15	1111	3	257	100000001
5	31	11111	4	65,537	10000000000000001

MERSENNE AND FERMAT NUMBERS have a simple appearance when written in binary notation. Although many Mersenne numbers are not primes (for example 15), there may be an infinite number of primes of this form. There may also be an infinite number of Fermat primes, but even the Fermat number for an n as small as 13 has not yet been tested.

3	11
6	110
9	1001
12	1100
15	1111
18	10010
21	10101
24	11000

27	11011
30	11110
33	100001
36	100100
39	100111
42	101010
45	101101
48	110000

INTEGERS DIVISIBLE BY THREE usually contain an even number of 1's when written in binary form. This observation led to the proof of a general theorem, described in the text.

$3^{(n-1)/2}$. Wells developed corresponding proofs for statements on integers divisible by five, seven and certain other numbers, although he found these theorems increasingly harder to prove.

By now quite a few problems in number theory have been studied experimentally on computers. Not all of this work is restricted to tables, special examples and sundry curiosities. D. H. Lehmer of the University of California at Berkeley has made unusually effective use of the computer in number theory. With its help he has recently obtained several general theorems. Essentially what he has done is to reduce general statements to the examination of a large number of special cases. The number of cases was so large that it would have been impracticable, if not impossible, to go through them by hand computation. With the help of the computer, however, Lehmer and his associates were able to determine all exceptions explicitly and thereby discover the theorem that was valid for all other cases. Unfortunately Lehmer's interesting work is at a difficult mathematical level and to describe it would take us far afield.

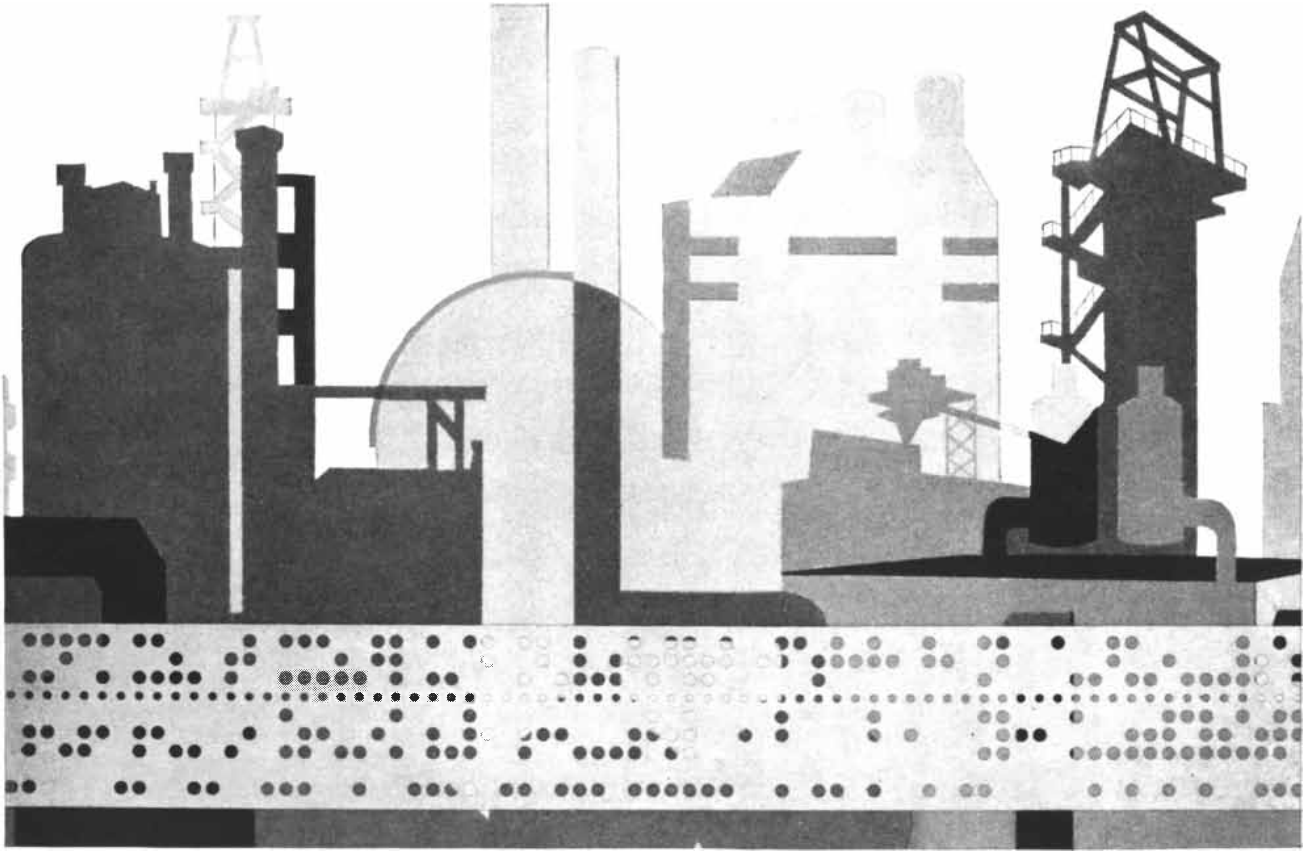
It must be emphasized that Lehmer's theorems were not proved entirely by machine. The machine was instrumental in enabling him to obtain the proof. This is quite different from having a program that can guide a computer to produce a complete formal proof of a mathematical statement. Such a program, however, is not beyond the realm of possibility. The computer can operate not only with numbers but also with the symbols needed to perform logical operations. Thus it can execute simple orders corresponding to the basic "Boolean" operations. These are essentially the Aristotelian expressions of "and," "or" and "not." Under a set of instructions the computer can follow such orders in a prescribed sequence and explore a labyrinth of possibilities, choosing among the possible alternatives the ones that satisfy, at any moment, the result of previous computations.

With such techniques it has been possible to program a computer to find proofs of elementary theorems in Euclid's geometry. Some of these efforts, particularly those pursued at the International Business Machines Research Center, have been quite successful. Other programs have enabled the computer to find proofs of simple facts of

1,2	1,3	1,4	1,5	1,6	1,7
	2,3	2,4	2,5	2,6	2,7
		3,4	3,5	3,6	3,7
			4,5	4,6	4,7
				5,6	5,7
					6,7

1,2,3	1,3	1,4,5,	1,5	1,6,7	1,7
	2,3	2,4,6	2,5,7	2,6	2,7
		3,4,7	3,5,6	3,6	3,7
			4,5	4,6	4,7
				5,6	5,7
					6,7

STEINER PROBLEM poses this question: Given n objects, can they be arranged in a set of triplets so that every pair of objects appears once and only once in every triplet? The problem can be solved only when $n = 6k + 1$ or $6k + 3$, in which k can be any integer. One solution for $k = 1$, in which case $n = 7$, is shown here. The table at left lists all possible pairs of seven objects. The table at right shows seven triplets that contain all pairs only once. The 21 digits in these triplets can be regrouped into other triplets.



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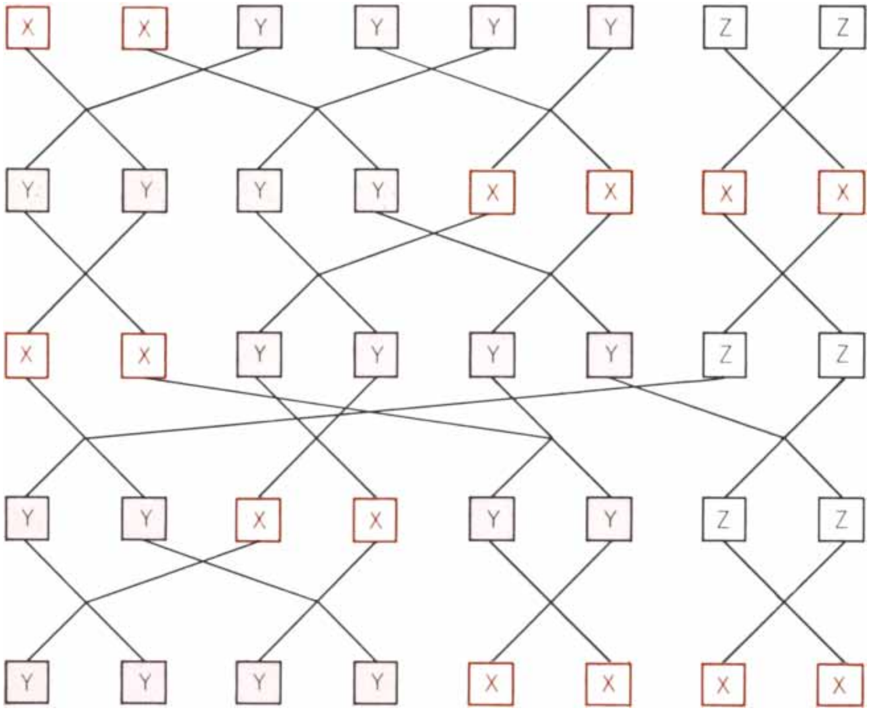


GENEALOGICAL "TREES" raise many interesting combinatorial questions. In the simple case shown here individuals of three different colors mate in pairs. Strictly, x, y and z specify the fraction of each color in each generation, but here they also identify color type. Each mating produces a pair of offspring and the color of the offspring is uniquely determined by the colors of the parents according to a fixed rule. (For example, 2 y's or 2 z's produce 2 x's.) Assuming an initial population containing hundreds of members, one might ask such questions as these: Given an individual in the fifth generation, how many different ancestors does he have in, say, the first generation? What are the proportions of x's, y's and z's among the ancestors of a given individual in the nth generation?

projective geometry. I have no doubt that these efforts mark only the beginnings; the future role of computers in dealing with the "effective" parts of mathematics will be much larger.

I shall now turn from the study of integers to combinatorial analysis and discuss some of the uses of the computer in this field. Very briefly, combinatorial analysis deals with the properties of arrangements and patterns defined by means of a finite class of "points." Familiar examples are the problems on permutations and combinations studied in high school algebra. In a typical case one starts with a finite set of n points and assumes certain given, or prescribed, relations between any two of them or, more generally, among any k of them. One may then wish to enumerate the number of all possible structures that are related in the prescribed way, or one may want to know the number of equivalent structures. In some cases one may consider the finite set of given objects to be transformations of a set on itself. In the broadest sense one could say that combinatorial analysis deals with relations and patterns, their classification and morphology. In this field too electronic computers have proved to be extremely useful. Here are some examples.

Consider the well-known problem of placing eight queens on a chessboard in such a way that no one of them attacks another. For an ordinary 8×8 chessboard there are only 12 fundamentally different solutions. The mathematician would like to know in how many different ways the problem can be solved for n queens on an $n \times n$ board. Such enumeration problems are in general difficult but computer studies can assist in their solution.

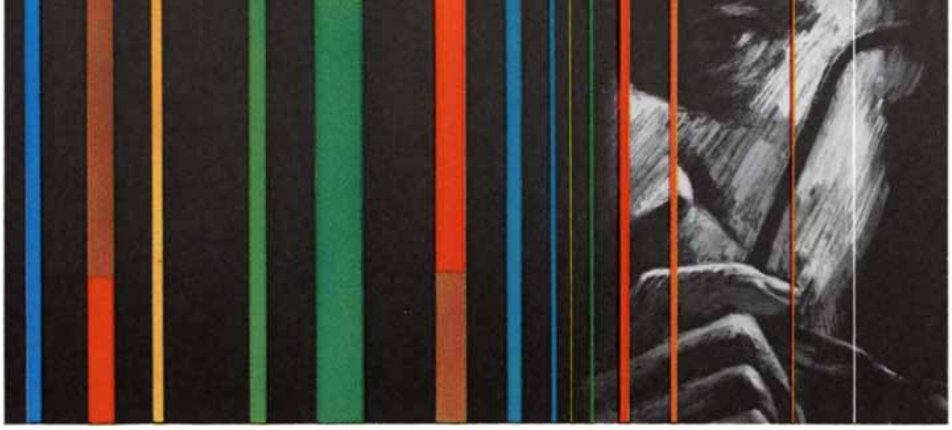
The following problem was first proposed in the 19th century by the Swiss mathematician Jakob Steiner: Given n objects, can one arrange them in a set of triplets in such a way that every pair of objects appears once and only once in a triplet? If n is five, for example, there are 10 possible pairs of five objects, but a little experimentation will show that there is no way to put them all in triplets without repeating some of the pairs. The problem can be solved only when $n = 6k + 1$ or $6k + 3$, in which k is any integer. The solution for $k = 1$ (in which case $n = 7$) is shown at the bottom of page 208. The number of triplets in the solution is seven. In how many ways can the problem be solved? Again, the computer is very useful when k is a large number.

The shortest-route problem, often called the traveling-salesman problem, is another familiar one in combinatorics. Given are the positions of n points, either in a plane or in space. The problem is to connect all the points so that the total route between them is as short as possible. Another version of this problem is to find the route through a network of points (without necessarily touching all the points) that would take the minimum time to traverse [see "Control Theory," page 194]. These problems differ from the two preceding ones in that they necessitate finding a method, or recipe, for constructing the minimum route. Strictly speaking, therefore, they are problems in "meta-combinatorics." This term signifies that a precise formulation of the problem requires a definition of what one means by a recipe for construction. Such a definition is possible, and precise formulations can be made. When the n points are distributed in a multidimensional space, the problem can hardly be tackled without a computer.

A final example of combinatorics can be expressed as a problem in genealogy. Assume, for the sake of simplicity, that a population consists of many individuals who combine at random, and that each pair produces, after a certain time, another pair. Let the process continue through many generations and assume that the production of offspring takes place at the same time for all parents in each generation. Many interesting questions of combinatorial character arise immediately.

For instance, given an individual in the 15th generation of this process, how many different ancestors does he have in, say, the ninth generation? Since this is six generations back it is obvious that the maximum number of different ancestors is 2^6, but this assumes no kinship between any of the ancestors. As in human genealogy there is a certain probability that kinship exists and that the actual number is smaller than 2^6. What is the probability of finding various smaller numbers?

Suppose the original population consists of two classes (that is, each individual has one or the other of two characteristics); how are these classes mixed in the course of many generations? In other words, considering any individual



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in the nth generation, one would like to know the proportion of the two characteristics among all his ancestors.

Let us now make a slightly more realistic assumption. Consider the process as before but with the restriction removed that all offspring appear at the same time from parents of the same age. Assume instead that the production of the new generation is spread over a finite period of time according to a specific probability distribution. After this process has continued for some time the individuals of the most recent generation will be, so to speak, of different generations. A process of this kind actually occurs in human populations because mothers tend to be younger, on the average, than fathers. Therefore going back, say, 10 generations through the chain of mothers yields a smaller number of total years than going back through the chain of 10 fathers. It becomes a complex combinatorial problem to calculate the average number of generations represented in the genealogical history of each individual after many years have elapsed from time zero. This and many similar questions are difficult to treat analytically. By imitating the process on a computer, however, it is easy to obtain data that throw some light on the matter.

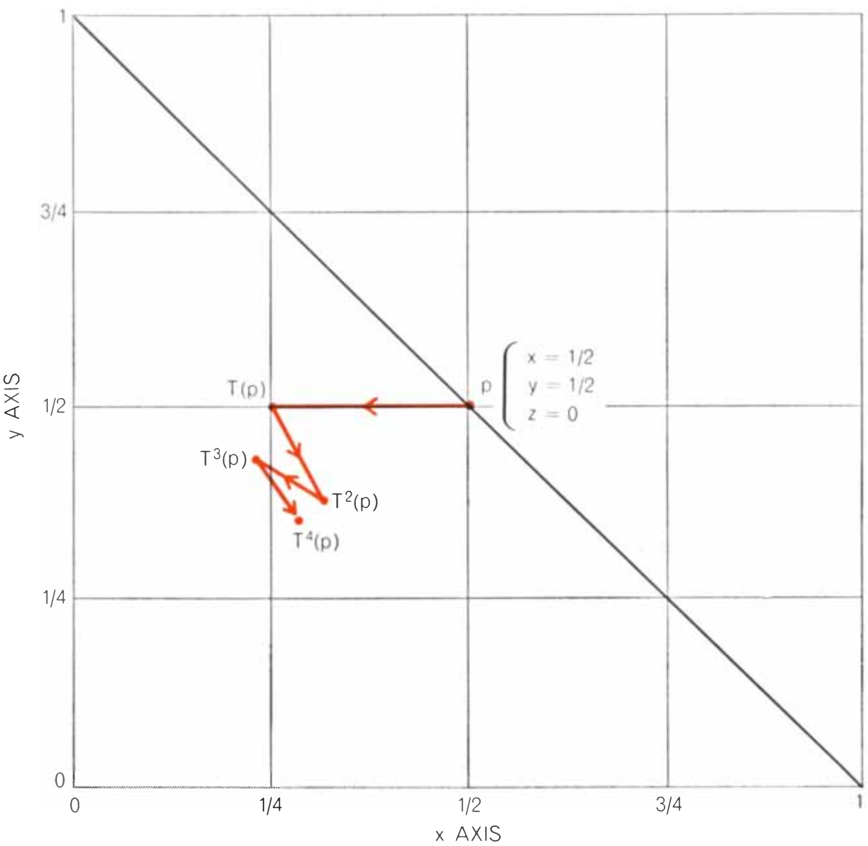
The last mathematical area I should like to discuss in connection with computers is the rather broad but little-explored one of nonlinearity. A linear function of one variable has the form $x' = ax + b$, where a and b are constants. Functions and transformations of this form are the simplest ones mathematically, and they occur extensively in the natural sciences and in technology. For example, quantum theory employs linear mathematics, although there are now indications that future understanding of nuclear and subnuclear phenomena will require nonlinear theories. In many physical theories, such as hydrodynamics, the equations are nonlinear from the outset.

The simplest nonlinear functions are quadratic; for one variable such functions have the form $y = ax^2 + bx + c$, where a, b and c are constants. It may surprise nonmathematical readers how little is known about the properties of such nonlinear functions and transformations. Some of the simplest questions concerning their properties remain unanswered.

As an example, mathematicians would like to learn more about the behavior of nonlinear functions when subjected to the process known as iteration. This simply means repeated application of the function (or transformation) to some starting value. For instance, if the point described by a function is the square root of x, the iteration would be the square root of the square root of x; each succeeding iteration would consist of again taking the square root.

A transformation given by two functions containing two variables each defines a point on a plane; its iteration gives rise to successive points, or "images" [*see illustration below*]. Finding the properties of the sequence of iterated images of a single point, when described by a nonlinear function, is in general difficult. Present techniques of analysis are inadequate to unravel the behavior of these quite simply defined transformations.

Here again empirical work with the



$$x' = y^2 + z^2 \qquad y' = 2xy + 2xz \qquad z = x^2 + 2yz$$

INITIAL POINT p	FIRST ITERATION T(p)	SECOND ITERATION $T^2(p)$	THIRD ITERATION $T^3(p)$	FOURTH ITERATION $T^4(p)$
x = 1/2	x′ = 1/4	x″ = 5/16	x‴ = 61/256 = .238	x⁗ = .295
y = 1/2	y′ = 2/4	y″ = 6/16	y‴ = 110/256 = .430	y⁗ = .363
z = 0	z′ = 1/4	z″ = 5/16	z‴ = 85/256 = .332	z⁗ = .342
SUM 1	1	1	1 = 1.000	1.000

PROCESS OF ITERATION involves repeated application of a function (or transformation) to an initial value or a point. Here three equations containing three variables define a point in a plane. Iteration gives rise to successive points, or "images." Because the three variables always add up to 1, only two variables (say x and y) need be plotted. The first iteration, $T(p)$, is obtained by inserting the initial values of x, y and z (½, ½, 0) in the three equations. The new values, x', y', z' (¼, ½, ¼), are then inserted to produce the second iteration, $T^2(p)$, and so on. Computers can quickly compute and display thousands of iterations of a point so that their behavior can be studied (*see examples on next page*).

computer can be of great help, particularly if the computer is equipped to display visually the location of many iterated points on the face of an oscilloscope. MANIAC II at Los Alamos has been equipped in this way and enables us to see at a glance the results of hundreds of iterations.

In examining such displays the mathematician is curious to learn whether or not the succession of iterated images converge to a single location, or "fixed point." Frequently the images do not converge but jump around in what appears to be a haphazard fashion—when they are viewed one by one. But if hundreds of images are examined, it may be seen that they converge to curves that are often most unexpected and peculiar, as illustrated in the four oscilloscope traces below. Such empirical work has led my associates and me to some general conjectures and to the finding of some new properties of nonlinear transformations.

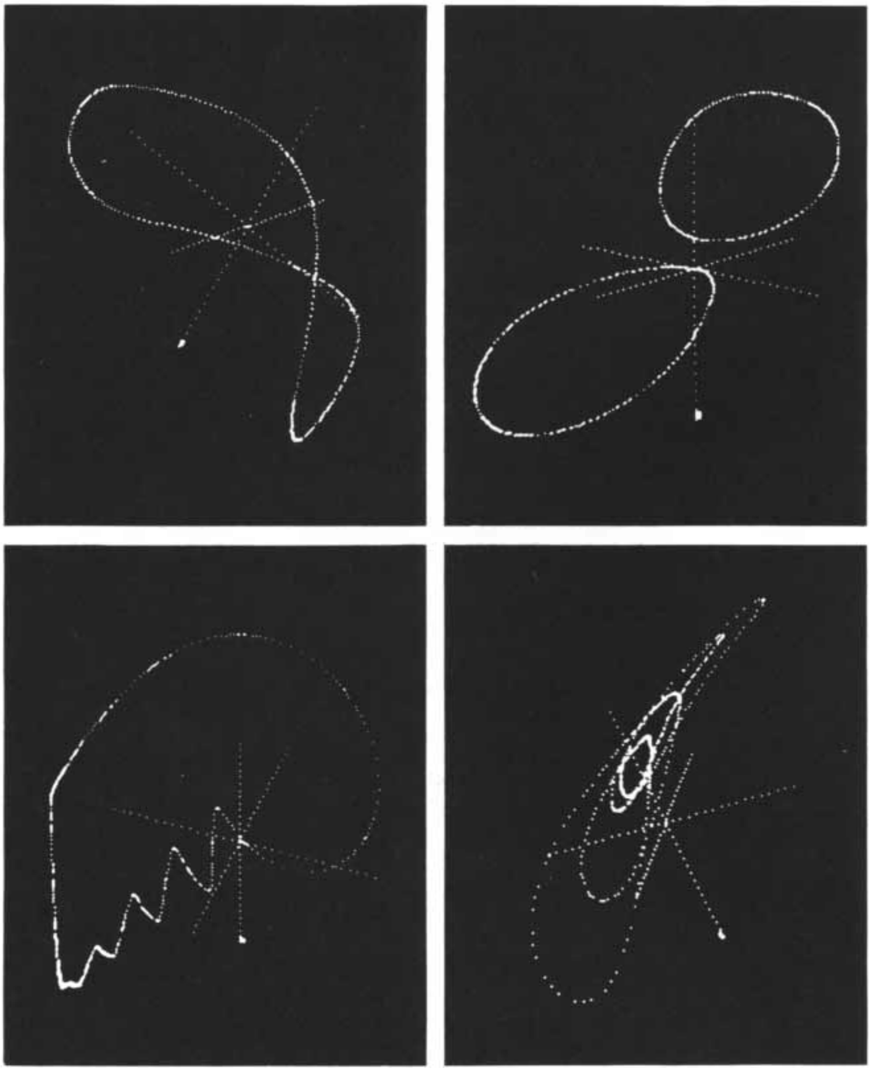
What are the obvious desiderata that would make the electronic computer an even more valuable tool than it is today? One important need is the ability to handle a broader range of logical operations. As I have noted, the simplest operations of logic, the Boolean operations, have been incorporated in electronic computers from the outset. In order to encompass more of contemporary mathematics the computer needs a "universal quantifier" and an "existential quantifier." The universal quantifier is required to express the statement one sees so frequently in mathematical papers: "*For all* x such and such holds." The existential quantifier is needed to express another common statement: "*There exists* an x so that such and such is true." If one could add these two quantifiers to the Boolean operations, one could formulate for computer examination most of traditional and much of modern mathematics. Unfortunately there is no good computer program that will manipulate the concepts "for all" and "there exists."

One can take for granted that there will be continued increases in processing speed and in memory capacity. There will be more fundamental developments too. Present computers operate in a linear sequence: they do one thing at a time. It is a challenge to design a machine more on the model of the animal nervous system, which can carry out many operations simultaneously. Indeed, plans exist for machines in which arithmetical operations would proceed simultaneously in different locations.

A multitrack machine would be of great value in the Monte Carlo method. The task of the machine is to compute individual histories of fictitious particles, and in many problems the fates of the particles are independent of one another. This means that they could be computed in parallel rather than in series. Moreover, it is not necessary that the computations be carried out to the many decimal places provided by present high-speed machines; an accuracy of four or five digits would often be enough. Thus it would be valuable to have a machine that could compute hundreds of histories simultaneously with only moderate accuracy. There are many other cases where a machine of such design would be efficient.

Further development is also desirable in facilitating the ease of transaction between the computer and its operator. At present it is difficult to change the course of a calculation as partial results become available. If access to the machine were more flexible and if the problem could be studied visually during the course of its development, many mathematicians would find experimentation on the computer more congenial than they do today.

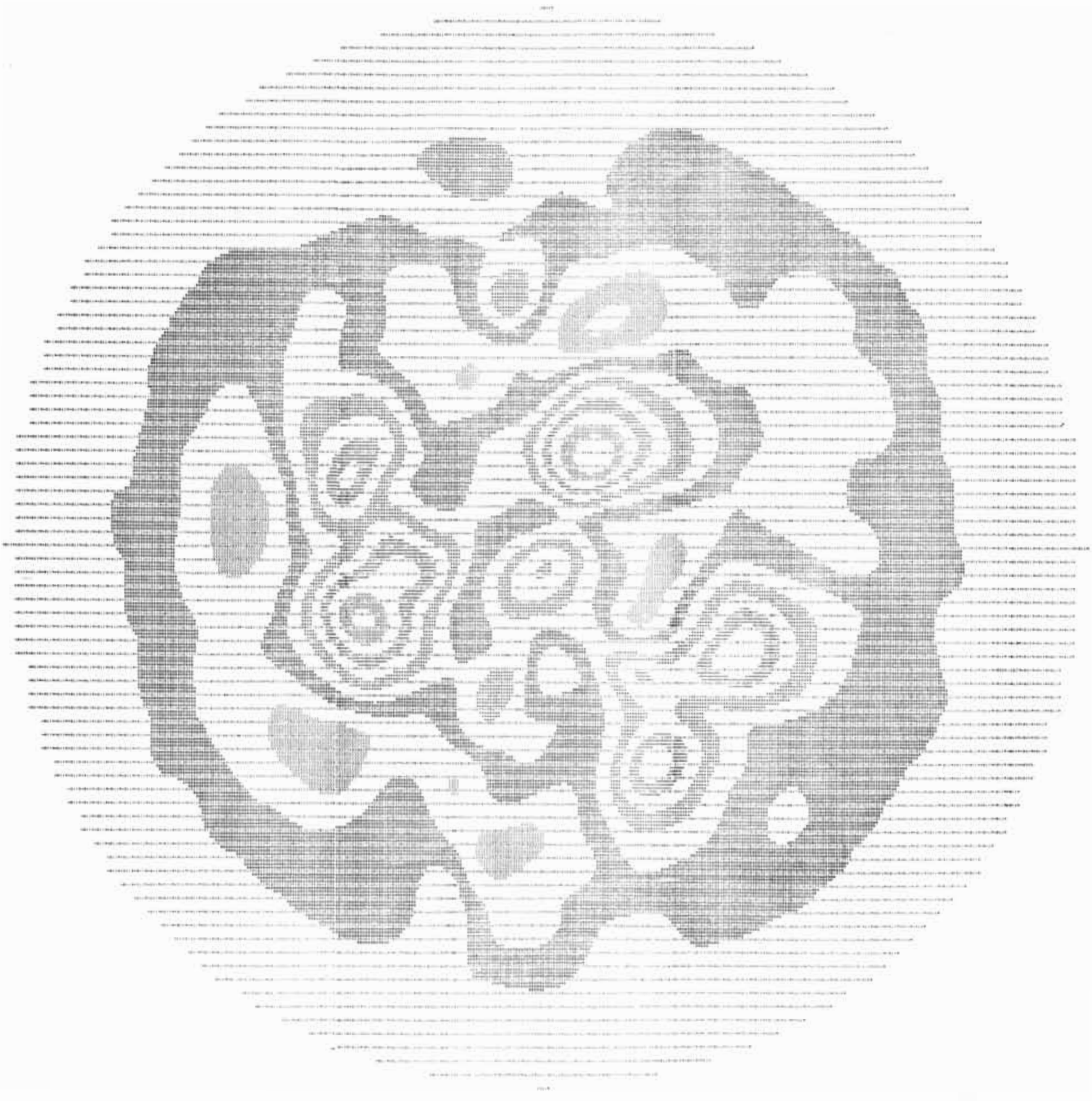
One can imagine new methods of calculation specifically adapted to the automatic computer. Thanks to the speed of the machine one will be able to explore,



ITERATIONS OF NONLINEAR TRANSFORMATIONS performed by high-speed Los Alamos computers are displayed on the face of an oscilloscope. The objective of this study by P. R. Stein and the author was to examine the asymptotic properties, or "limit sets," of iterates of certain nonlinear transformations of relatively simple form. These iterations are for sets of four functions containing four variables and therefore must be plotted in three dimensions; the straight dotted lines indicate the coordinate axes (*see two-dimensional plotting on preceding page*). The figure at top left is a twisted space curve. That at top right consists of two plane curves. The two bottom figures are more complicated.

almost palpably, so to say, geometrical configurations in spaces of more than three dimensions and one will be able to obtain, through practice, new intuitions. These will stimulate the mathematician working in topology and in the combinatorics of new mathematical objects. These objects may be ordinary integers, but integers far exceeding in size and number any now used for experimentation. One should also be able to develop mathematical expressions with many more existential quantifiers than are now employed in formal mathematical definitions. New games will be played on future machines; new objects and their motions will be considered in spaces now hard to visualize with our present experience, which is essentially limited to three dimensions.

The old philosophical question remains: Is mathematics largely a free creation of the human brain, or has the choice of definitions, axioms and problems been suggested largely by the external physical world? (I would include as part of the physical world the anatomy of the brain itself.) It is likely that work with electronic machines, in the course of the next decade or so, will shed some light on this question. Further insight may come from the study of similarities between the workings of the human nervous system and the organization of computers. There will be novel applications of mathematics in the biological sciences, and new problems in mathematics will be suggested by the study of living matter.



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MATHEMATICAL GAMES

Puns, palindromes and other word games that partake of the mathematical spirit

by Martin Gardner

Was I clever enough? Was I charming?
Did I make at least one good pun?
—JOHN UPDIKE,
Thoughts while Driving Home

Because every article in this issue is devoted to mathematics, it seems only charitable to center our attention this month on as unmathematical a topic as possible, without abandoning mathematics altogether. Word play—puns, anagrams, palindromes and so on—is certainly not discussed in any mathematics book, yet it has about it a quasi-mathematical air. Letters are symbols that combine according to rules to form words; words are symbols that combine according to rules to form sentences. Perhaps this combinatorial aspect is the reason so many mathematicians are addicted to language play.

The impulse to pun can persist even in the face of imminent death. On March 22, 1963, a murderer named Frederick Charles Wood was executed at Sing Sing. According to newspaper accounts, just before seating himself in the electric chair Wood said to those present: "I have a speech to make on an educational project. You will see the effect of electricity on Wood."

Less grim was the *New York Times* report a month later (April 28) that a gnu in the Chessington Zoo in England had bitten a zoo keeper. Odd, said the keeper, "most gnus are good gnus." I also find in my files an Associated Press dispatch from Des Moines, dated October 11, 1960, reporting that a perfume-dispensing machine in the women's lounge of a local hotel had failed to work. The management had hung a sign on it that read "Out of order." An unidentified patron, using lipstick, had crossed out the first "r" of "order."

The last is not strictly speaking a pun but rather a crude example of what word puzzlists call a deletion: the changing of one word into another by the removal of a letter. An amusing deletion story is told about Lord Kelvin, the British mathematician and physicist. He once put a sign on the door of a lecture hall stating that he would be unable to "meet my classes today." A student beheaded the word "classes" by crossing out the "c." Next day, eager to observe the professor's reaction, the students found that he had one-upped them by performing a second beheading.

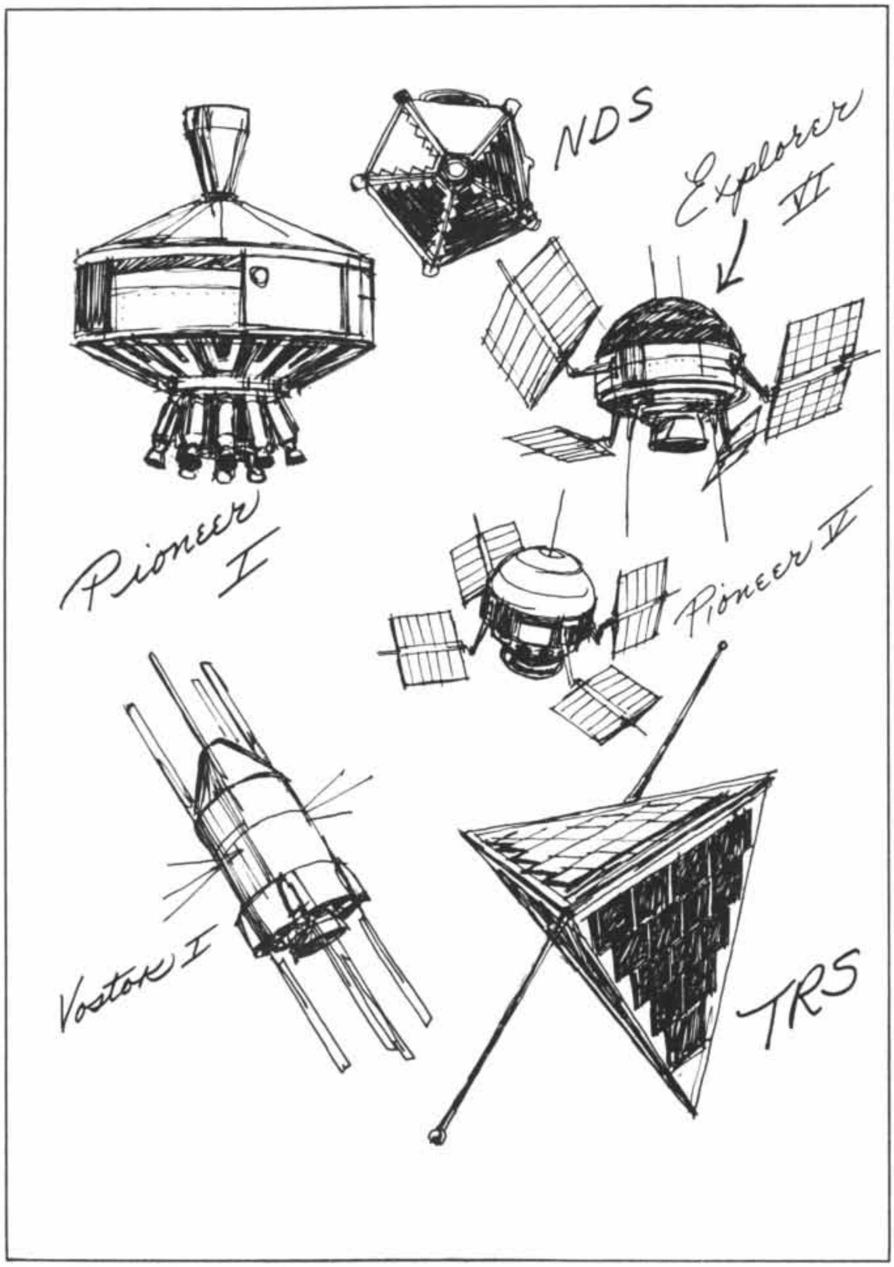
The following sentence is unusual: "Show this bold Prussian that praises slaughter, slaughter brings rout." If each word is beheaded, an entirely new sentence results. It is startling to learn that "startling" can be changed into eight other familiar words by successive deletions (from different places) of single letters. George Canning, an early-19th-century British statesman, wrote the following verse about a word that is subject to "curtailment," that is, a word that becomes a different word when its last letter is removed. Can you identify the word?

A word there is of plural number,
Foe to ease and tranquil slumber;
Any other word you take
And add an "s" will plural make.
But if you add an "s" to this,
So strange the metamorphosis,
Plural is plural now no more,
And sweet what bitter was before.

Both decapitation and curtailment are involved in the following old riddle:

From a number that's odd,
cut off the head,
It then will even be;
Its tail I pray now take away,
Your mother then you'll see.

It would be interesting to know how many technical books of recent years have messages concealed in the text by playful authors. One finds out about them by accident. Who would have guessed, for example, that *Transport Phenomena,* a 780-page textbook by R. Byron Bird, Warren E. Stewart and



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Edwin N. Lightfoot (published by John Wiley & Sons in 1960), had "On Wisconsin" hidden on page 712? (It is spelled by the first letters of each paragraph.) Or that the first letters of each sentence in the preface spell "This book is dedicated to O. A. Hougen"?

Sometimes word play enters a technical book fortuitously. Recently I had occasion to look up something in Rudolf Carnap's great work on semantics, *Meaning and Necessity*. On page 63 I came across a stretch of text in which the views of Black are sharply contrasted with those of White. Surely these were hypothetical individuals introduced to clarify an obscure point. No, on closer inspection they turned out to be the well-known philosophers Max Black and Morton White!

A classic instance of accidental word play is provided by the first (1819) edition of William Whewell's *Elementary Treatise on Mechanics*. On page 44 the text can be arranged in the following form:

There is no force, however great,
Can stretch a cord, however fine,
Into a horizontal line,
Which is accurately straight.

The buried poem was discovered by Adam Sedgwick, a Cambridge geologist, who recited it in an afterdinner speech. Whewell was not amused. He removed the poem by altering the lines in the book's next printing. Whewell actually published two books of serious poetry, but this unintended doggerel is the only "poem" by him that anyone now remembers.

If you keep your ears tuned, accidental meters turn up more often than you would expect. Max Beerbohm's eye caught the unintended beat in the following lines on the copyright page of the first English edition of one of his books:

London: John Lane, The Bodley Head
New York: Charles Scribner's Sons

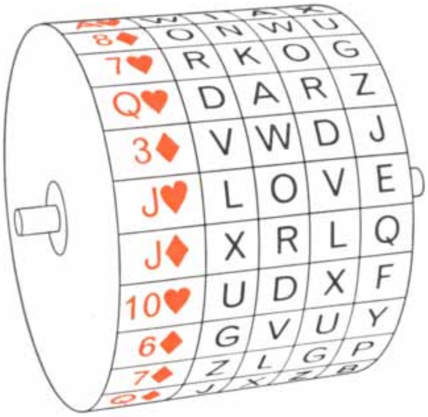
Beerbohm completed the quatrain by writing

This plain announcement, nicely read,
Iambically runs.

"Quintessential light verse," wrote John Updike, commenting recently on the above lines, "a twitting of the starkest prose into perfect form, a marriage of earth with light, and quite magical. Indeed, were I a high priest of literature, I would have this quatrain made

1				E											
2	A	B		C			D	E							
3															
4			M	N	O	P									
5					R	S	T	U							
6	U		O		I	E			A						
7		A		E		I	O	U			Y				
8		E		I		I		I		I		U		E	

Eight curious words



Gimmick for the "Think a word" trick

into an amulet and wear it about my neck, for luck."

The spoonerism, in which parts of two words (usually first syllables) are switched, continues to flourish as a popular form of wit. In 1960 Adlai Stevenson was campaigning in St. Paul, Minn., when the clergyman Norman Vincent Peale made some unfortunate political remarks. Stevenson told the press that he found St. Paul appealing and Peale appalling, surely one of the finest of all topical spoonerisms. In 1962, shortly after Rembrandt's painting "Aristotle Contemplating the Bust of Homer" had been bought by New York's Metropolitan Museum of Art for $2,300,000, it seems that Aristotle Onassis, the Greek shipping magnate, was shown Buster Keaton's house by a real estate agent. It was widely reported that a photograph in a Los Angeles newspaper was captioned "Aristotle Contemplating the Home of Buster," although I cannot vouch for it.

Ogden Nash's verse abounds in splendid spoonerisms:

. . . I am a conscientious man,
when I throw
rocks at sea birds
I leave no tern unstoned,
I am a meticulous man
and when I portray
baboons I leave no stern untoned.

No discussion of word play should fail to mention James Joyce. *Finnegans Wake* has, by a conservative estimate, 200 verbal plays per page, or more than 125,000 all together. The mathematical section of this book, pages 284 to 308 of the edition published by the Viking Press in 1947, contains hundreds of familiar mathematical terms, scrambled with metaphysics and sex. (The geometric diagram on page 293 is discussed mainly as a sex symbol.) The first footnote, "Dideney, Dadeney, Dudeney," refers to Henry Ernest Dudeney, the great English puzzle expert of Joyce's day. On page 302 "Smith-Jones-Orbison?" alludes to one of Dudeney's most popular puzzles, a logic problem involving three men named Smith, Jones and Robinson (see this department for February, 1959). Another of Dudeney's puzzles turns up in a footnote on page 299: "Pure chingchong idiotism with any way words all in one soluble. Gee each owe tea eye smells fish. That's U."

The puzzle: If you pronounce "gh" as in "tough," "o" as in "women" and "ti" as in "emotion," how do you pronounce "ghoti"? Was Joyce, in this footnote, speaking of the book itself and calling his reader a poor fish for biting the hook?

There are many references in *Finnegans Wake* to Lewis Carroll, who, as everyone knows, was a mathematician. In the mathematics section we read (page 294): "One of the most murmurable loose carollaries ever Ellis threw his cookingclass." (Need I point out that the last phrase puns on Alice *Through the Looking-Glass*?)

The following excerpt is from page 284: ". . . palls pell inhis heventh glike noughty times ∞, find, if you are not literally cooefficient, how minney combinaisies and permutandies can be played on the international surd! pthwndxrclzp!, hids cubid rute being extructed, taking anan illitterettes, ififif at a tom. Answers, (for teasers only)."

A partial explication: Pell was a mathematician for whom the Pellian equation was named, a number theorem often mentioned by Dudeney. "Heventh" is a compression of "seventh heaven." "Pthwndxrclzp" is one of the book's many thunderclaps. "Taking anan illitterettes, ififif at a tom" is, I suppose, "taking any letters, fifty at a time." "For teasers only" is a play on "for teachers only."

The pangram, an ancient form of word play, is an attempt to get the maximum number of different letters into a sentence of minimum length. The English mathematician Augustus De Morgan tells (in his *A Budget of Paradoxes*) of unsuccessful labors to write an intelligible sentence using every letter once and only once. "Pack my box with five dozen liquor jugs" gets all 26 letters into a 32-letter sentence, and "Waltz, nymph, for quick jigs vex Bud" cuts it to 28. Dmitri Borgmann of Oak Park, Ill., the country's leading authority on word play, has devised a number of

26-letter pangrams, but all require explanation. His best is "Cwm, fjord-bank glyphs vext quiz." A "cwm" is a circular valley, "quiz" is an 18th-century term for an eccentric, a "glyph" is a carved figure. The sentence thus states that an eccentric person was annoyed by carved figures on the bank of a fjord in a circular valley. Can any reader supply a better 26-letter pangram?

Another old and challenging word curiosity is the palindrome, a sentence that is spelled the same backward and forward. Borgmann's collection, covering all major languages, runs to several thousand. In my opinion the finest English palindrome continues to be "A man, a plan, a canal—Panama!" It has recently been attributed to James Thurber, but it was composed many years ago by Leigh Mercer of London, one of the greatest living palindromists. An unpublished Mercer palindrome, which is also something of a tongue twister, is "Top step's pup's pet spot."

Another Mercer palindrome, remarkable for both its length and naturalness, is "Straw? No, too stupid a fad. I put soot on warts." J. A. Lindon of Weybridge, England, is another master palindromist who turns them out by the hundreds. Who would suspect a palindrome if, in a novel, he came on the following Lindon sentence: "Norma is as selfless as I am, Ron." Lindon has also composed a large number of palindromes in which words rather than letters are the units. For instance: "So patient a doctor to try to doctor a patient so" and "Amusing is that company of fond people bores people fond of company that is amusing."

Composing anagrams on the names of friends or prominent people was once a fashionable literary sport. De Morgan tells of a friend who composed 800 anagrams on "Augustus De Morgan" (sample: "O Gus! Tug a mean surd!"). Lewis Carroll proudly recorded in his diary for November 25, 1868, that he had sent to a newspaper an anagram "which I thought out lying awake the other night: William Ewart Gladstone: Wilt tear down *all* images? I heard of another afterwards, made on the same name: 'I, wise Mr. G., want to lead all'—which is well answered by 'Disraeli: I lead, Sir!' " When Grover Cleveland was president, someone turned his name into "Govern, clever lad!" Theodore Roosevelt anagrams to "Hero told to oversee" and Dwight D. Eisenhower to "Wow! He's right indeed!" During the 1936 election, Borgmann also informs me, the letters of Franklin Delano Roosevelt's name were permuted—by a Republican, no doubt—to "Vote for Landon ere all sink!" It was said during this campaign that the Republicans avoided picking Styles Bridges, at that time governor of New Hampshire, for Landon's running mate for fear the Democrats would go about chanting "Landon-Bridges falling down."

With an election now looming, what can readers do with the full names of the two candidates Lyndon Baines Johnson and Barry Morris Goldwater? I will be unable to reply to letters, but the best results will be reported in this department in December.

For less ambitious readers the illustration on page 220 presents eight remarkable English words, the missing letters to be supplied. All letters omitted from the first word are consonants. The second word contains the first five letters of the alphabet in order. The third word can be typed by using only the top row of keys on a standard typewriter. (The letters of this row, left to right, are *QWERTYUIOP*.) The fourth and fifth words contain four letters in adjacent alphabetical order. The sixth word contains the five vowels in reverse order, the seventh the five vowels plus *Y* in the usual order. In the last word consonants and vowels alternate. These eight words, and the two that answer the two rhymed riddles, will be supplied next month.

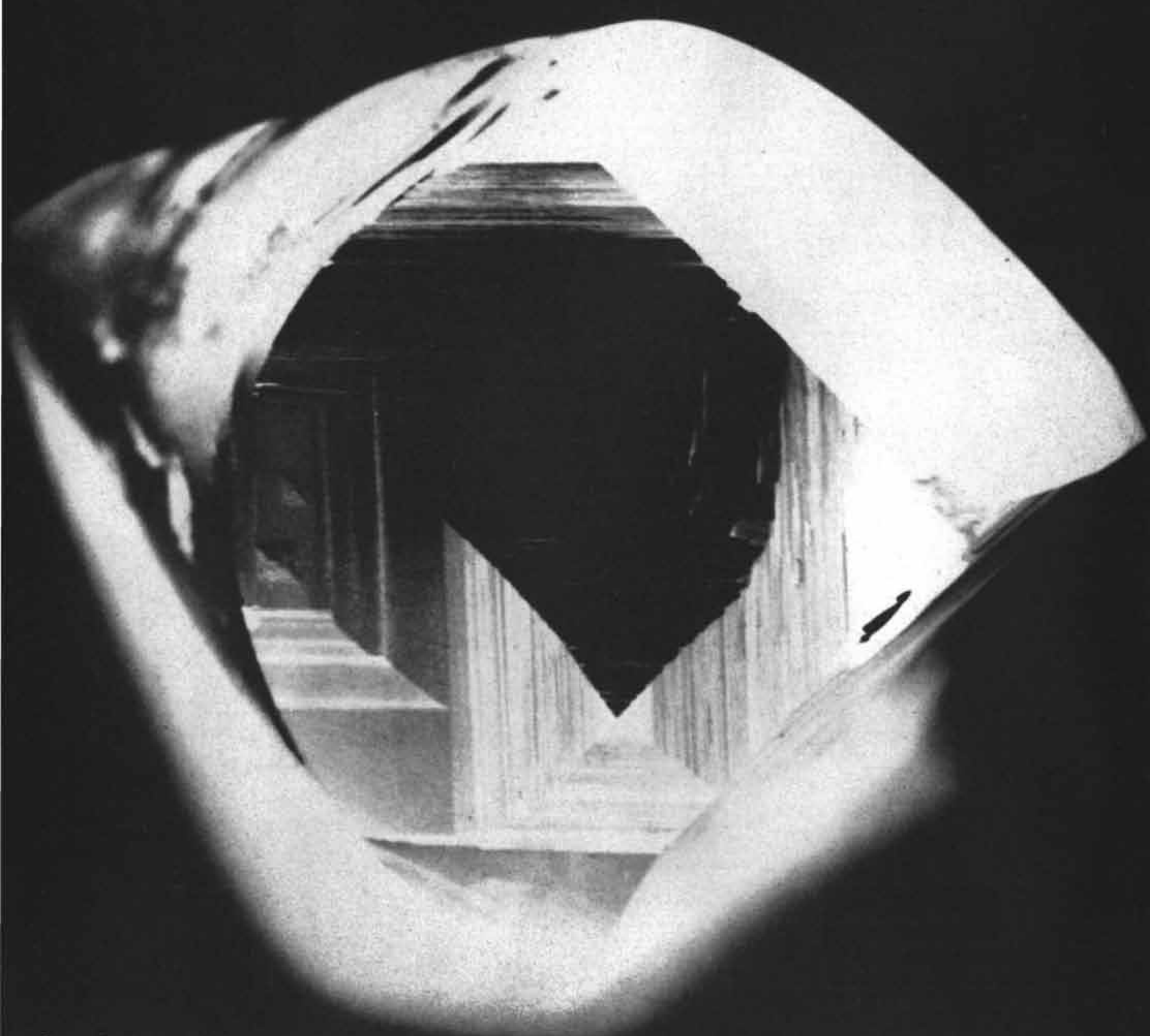
In the illustration for the "Think a word" trick described last month the four-letter word being spelled is "love" and the thought-of card is the jack of hearts. To determine the word the magician uses a gimmick: a cylinder of five disks that rotate around a pin as shown in the illustration on page 222. The 26 red cards are in the same order around the rim of the first disk as they are on the spelling wheel, and the 26 letters on each of the other disks are in the same order as those that surround the spelling wheel.

When the first letter of the word is spelled, the magician glances at the wheel and notes the letter opposite any card whatever, say the ace of hearts. As soon as his back is turned he rotates the second disk of his gimmick until this letter touches the ace of hearts. On his second glance at the wheel he notes the new letter opposite the ace of hearts. When his back is turned again, he adjusts the third disk accordingly. Similarly for the remaining two letters. In other words, the performer himself picks a card and uses it to spell four letters. He adjusts his dials so that his card and these four letters are in line. Then he turns the entire cylinder until he sees a four-letter word. It will be the word the spectator spelled. There is, of course, a chance that more than one word will turn up, but the odds are heavily against it. If it should happen, the magician simply makes more than one guess.

The gimmick can be made small enough to keep concealed in one hand. A similar gimmick can be made by mounting four concentric circles of graduated size on a square of cardboard. This can be kept in the performer's inside coat pocket, to be pulled out and secretly adjusted each time his back is turned. By adding more disks to the cylinder, or circles to the cardboard, one can do the trick with longer words. If the word is long, one is often able to spot the only possible combination of letters before the spelling is completed and so guess the word. In such cases a final look at the wheel will verify the guess, then the performer can proceed to name the word without turning his back again.

Some magicians omit the card symbols entirely from the gimmick. This has no effect on their ability to guess the word, and if someone asks them if they also know the selected card, they can answer, in complete honesty, that they haven't the slightest idea what it is!

Readers may be interested to learn that an elliptical pool table is now on sale. A full-page advertisement in *The New York Times* for July 1 announced that the game would be introduced, on the following day, by Broadway stars Joanne Woodward and Paul Newman at Stern's department store. Elliptipool, as it is called, sells for $125 and is the patented invention of Arthur Frigo of Torrington, Conn., a student at Union College in Schenectady. Because the table's one pocket is exactly at one of the two foci of the ellipse, a variety of weird cushion shots can be made with ease (see this department's discussion of elliptical billiards in February, 1961). Lewis Carroll once had built for him a circular billiard table. The 11th edition of the *Encyclopædia Britannica* in its article on billiards has a footnote that reads: "In 1907 an oval table was introduced in England by way of a change." Neither table had a pocket, however. A design patent (198,571) was issued in July to Edwin E. Robinson of Pacifica, Calif., for a circular pool table with four pockets.



Crystal Surface

THE AMATEUR SCIENTIST

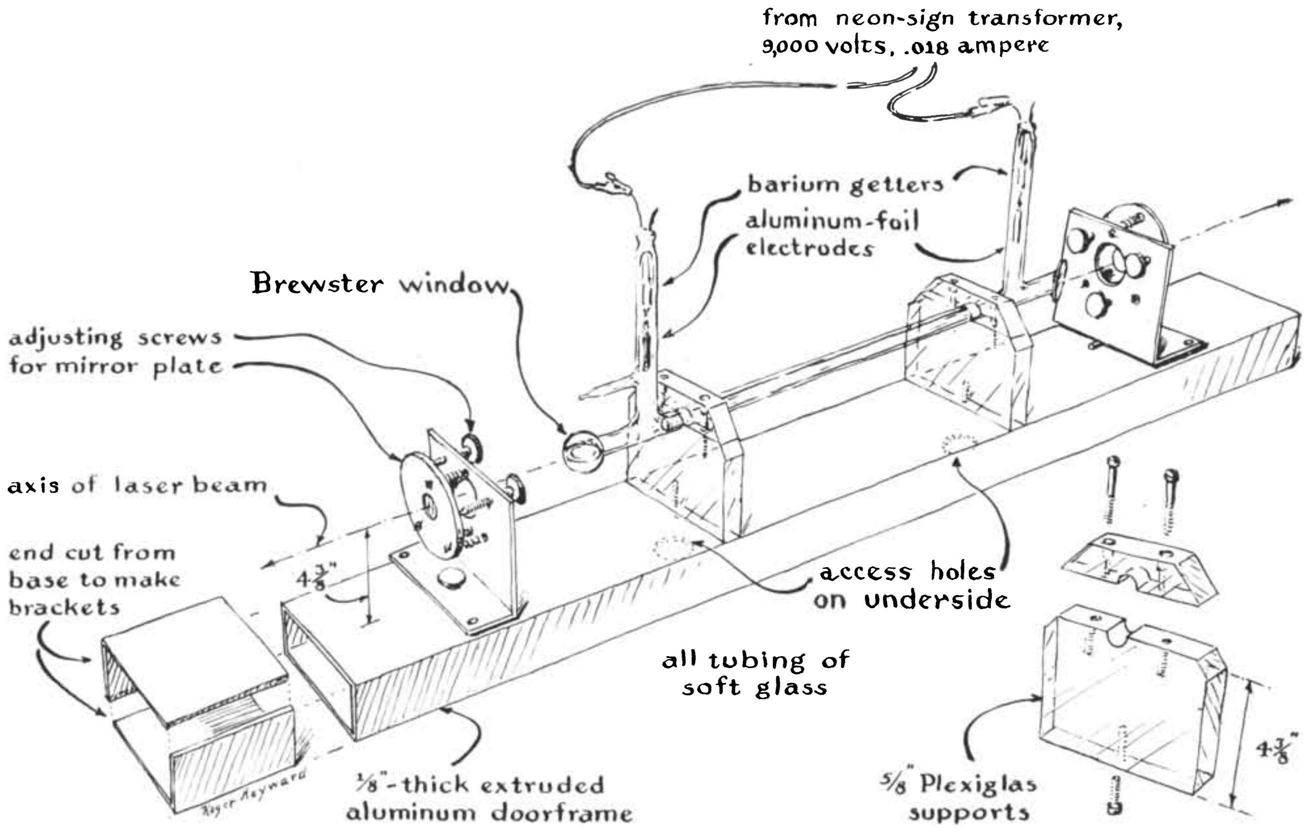
How a persevering amateur can build a gas laser in the home

Conducted by C. L. Stong

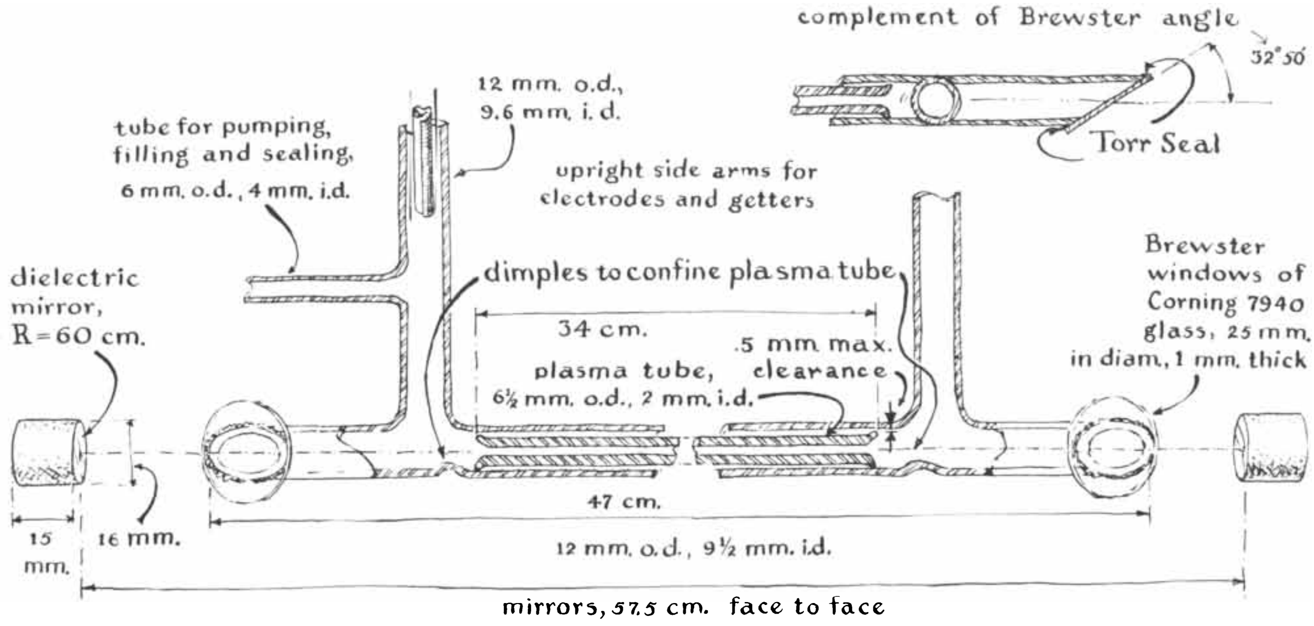
Few devices in the modern physics laboratory present a more deceptive appearance of simplicity than the helium-neon laser, a device with many exciting prospects [see "Advances in Optical Masers," by Arthur L. Schawlow; SCIENTIFIC AMERICAN, July, 1963]. The apparatus seems to consist merely of a gas-discharge tube that looks much like the letter "I" in a neon sign; at the ends of the tube are flat windows that face a pair of small mirrors. Yet when power is applied, the device emits as many as six separate beams of intense coherent light.

What accounts for this remarkable performance? You can discover the answer by the pleasant if somewhat slow method of undertaking to build a laser at home, as I did earlier this year. Not only will you learn what makes the laser tick but also, as a bonus, you may encounter some fascinating properties of light that you have previously overlooked. If you are as inexperienced as I was, however, you may not find the project as easy or as inexpensive as some that have been discussed in these columns. I can promise that it will exercise your talents for such diverse arts as blowing glass, fabricating small parts, maintaining scrupulous cleanliness in the workshop and operating a high-vacuum system. The cost will vary inversely in proportion to your capacity for improvisation, but you can expect it to exceed $100.

The gas laser requires special structures, the need for which arises because the device is an extremely poor amplifier, at least by electronic standards. In the visible region of the spectrum it usually has a maximum gain of less than 10 percent. Much of the input power is wasted by excited atoms of ionized gas that emit light in random directions. Some emission, however, travels along the axis of the gas-discharge tube and is reflected back and forth between the mirrors. During each transit this oscillating light stimulates still other excited atoms to emit energy that falls into lockstep with the same waves that triggered the emission. The stimulated emission thus increases the intensity of the light,



A helium-neon laser designed for amateur construction



Details of the laser tube assembly

but only by a few percent on each pass.

During each transit the light must make its way through the end windows to the mirrors and back again. These surfaces are obstacles that can introduce major losses both by absorption and by reflection, as is evident from what happens with an ordinary glass window. When light falls at right angles on such a window, about 4 percent of the energy is reflected back to the source by each of the window's two surfaces. A lesser amount is also absorbed, being transformed into heat by the glass. In an imperfectly aligned window of the laser these losses combine to reduce the intensity of a ray more than 16 percent in the course of a single pass, more than is gained from the stimulated emission. Perfect alignment of a laser window is impossible.

The inventors of the gas laser at the Bell Telephone Laboratories found a solution for the problem of reflection losses in the century-old work of Sir David Brewster, the Scottish physicist who discovered that light is strongly polarized when it falls at a certain critical angle on a sheet of glass or some other transparent medium, and that no reflection occurs in the case of light waves so polarized that they vibrate only in the plane of incidence. The effect is observed when the tangent of the angle between the surface of the window and a line drawn perpendicularly to the rays equals the refractive index of the glass.

To cope with the second source of loss at the laser windows—the conversion of light into heat—the designers simply substituted quartz for glass. At a wavelength of 6,328 angstrom units, which is the wavelength of the light in a helium-neon laser, the heat loss in quartz is slight compared with that in glass. The refractive index of quartz is 1.54, equivalent to an angle of 57 degrees. Quartz plates installed in the laser at this angle are called Brewster windows. If the Brewster windows at each end of the laser envelope are in the same plane of polarization, the transmission of light through the assembly approaches 100 percent.

The designers found an equally ingenious solution for the problem of losses at the mirrors. Freshly silvered glass reflects as much as 96 percent of the incident light, but as the metal film tarnishes in the atmosphere its performance falls sharply. Aluminum, the next best metallic coating, reflects at most only 92 percent of the incident light.

To achieve higher performance the designers of the laser abandoned conventional reflectors in favor of dielectric mirrors, which are mirrors coated with several nonconducting films. For high efficiency such mirrors depend on interference among the light waves reflected by the multilayered films, which are composed of transparent substances such as sulfides and fluorides. The films can be designed either to suppress reflection, as they do in the familiar coating on the lenses of cameras, or to enhance it. If the refractive index of a transparent film a quarter of a light wave thick is lower than that of the glass on which it is deposited, a wave of light reflected by the glass arrives at the surface of the film 180 degrees out of step with a wave reflected by the film. The crest of one wave falls in step with the trough of the other and the two cancel. If the refractive index of the film is higher than that of the glass, the crests and troughs of the two combine to increase the reflectivity.

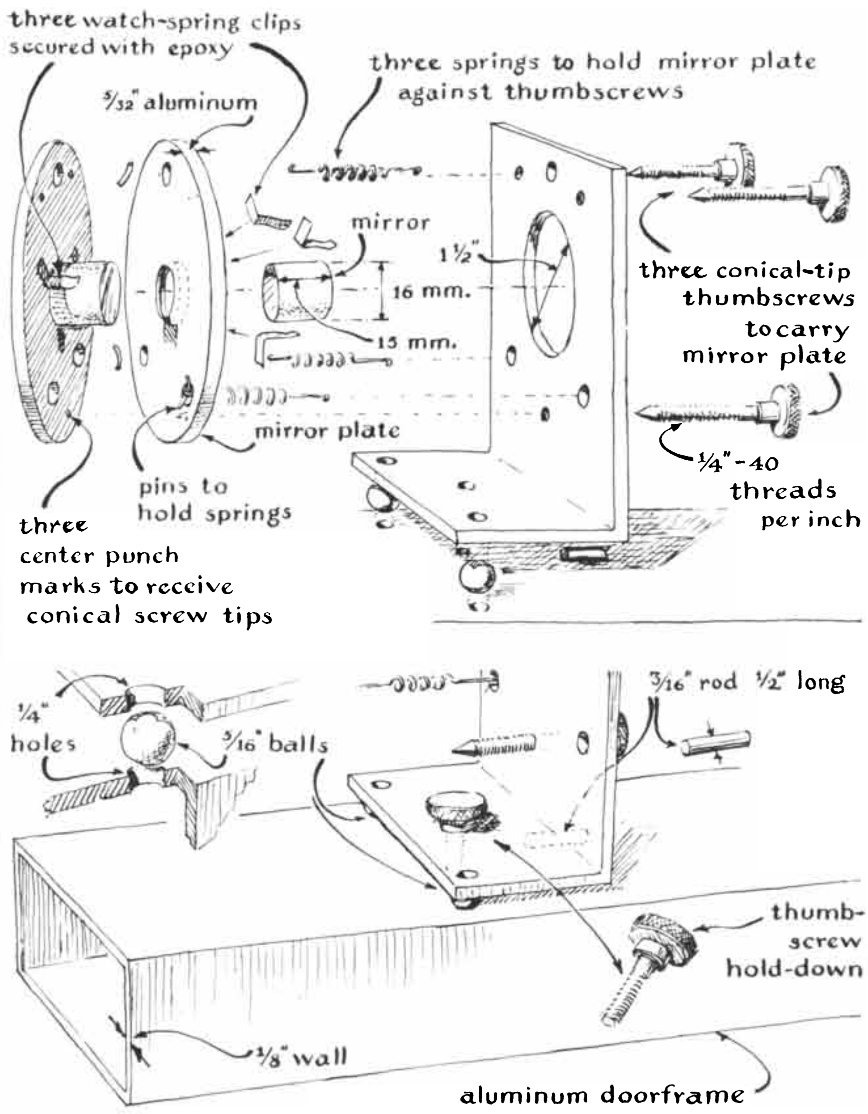
A second film of lower refractive index than the glass, when applied over the first film, reflects waves of opposite phase with respect to those reflected from the glass. The second film is located a full half-wavelength away from the glass, however, which precisely compensates for the difference in phase. The waves therefore again interfere constructively to increase the reflectivity. The application of a third film reduces the reflectivity somewhat but its effect is more than compensated by the fourth film, and so on. The dielectric mirrors used in lasers employ between 13 and 27 films, and the reflectivity of such a mirror approaches 100 percent. Incidentally, the manufacture of dielectric mirrors requires facilities and techniques that are normally beyond the reach of amateurs. These components, like the gas mixture for the laser, must be bought. A list of suppliers appears at the end of this article.

When the combined losses have been minimized by suitable techniques, the intensity of the reverberating light increases, but not without limit. The growing electromagnetic field between the mirrors interferes increasingly with the

number of excited atoms that respond to the influence of the field and so are stimulated to emit energy. A delicate balance is eventually achieved between the losses and the gain. At this point energy gained by stimulated emission precisely equals the combined losses, including the portion that escapes from the apparatus in the output beams.

In addition to these losses, a falling off in gain can occur inside the tube assembly. The assembly consists of a glass envelope that supports the windows, a pair of electrodes and the plasma tube—a slender tube of small bore in which the discharge occurs. Internal losses become serious if the helium-neon mixture is contaminated by even a trace of foreign gases such as oxygen, nitrogen and carbon dioxide, or if the pressure of the gas is not maintained within certain limits. The problem of contamination is met largely by removing from the tube all unwanted gases and all substances that can release vapors. The inner parts of the device must be immaculate. Even the faintest smudge from a fingertip can release an astonishing amount of vapor. The clean tube is partly cleared of unwanted gases by the vacuum pumps. It is evacuated to a pressure of at least 10^{-5} torr. (A torr is the pressure that will support a column of mercury one millimeter in height.) The remaining contamination is then immobilized by firing a "getter," an electrically heated crucible inside the tube from which vaporized barium condenses on the glass walls. The barium unites chemically with most elements other than the inert gases.

The optimum pressure at which stimulated emission occurs varies inversely with the diameter of the gas-discharge



Parts and arrangement of fixture to support mirror cell

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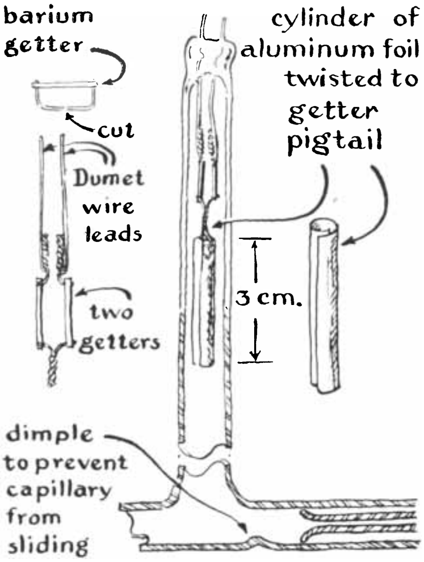


The getter-electrode assembly

tube. The pressure in torrs is equal to 3.6 divided by the inside diameter of the tube in millimeters. The laser will continue to operate at diminishing intensity up to about twice the optimum pressure and down to about half of it. Advantage is taken of this fact to extend the service life of laser tubes by overfilling them by 50 percent, because for reasons not fully understood the gas pressure drops slowly as the laser operates. Lasers will operate best on gas ratios of seven parts of helium to one of neon, but the tubes can be filled with a nine-to-one mixture.

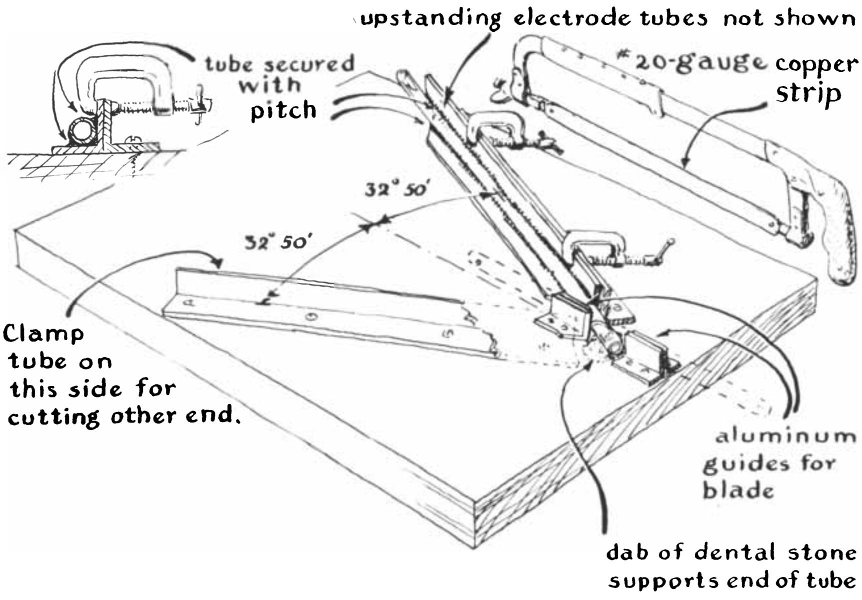
The range of tube diameters that can be used in the laser is restricted by the nature of ionized gas and by mechanical considerations. At pressures much lower than .5 torr the electrons acquire enough energy to damage the glass envelope by impact and to erode the electrodes. The metal vaporized in this way condenses on the envelope and in the process buries gas atoms, lowering the pressure still more. Thus a runaway effect develops that causes the tube to fail. An envelope that contains a plasma tube six millimeters in diameter ought to be filled, according to the formula given above, to a pressure of 3.6/6, or .6 torr. With a laser of the type I built the life of a tube filled to this marginal pressure would be impractically short. I use a plasma tube with a diameter of two millimeters, and I overfill it to a pressure of 2.7 torrs. Tube diameters of less than about one millimeter become awkward to align and have other drawbacks.

In general the output of a laser increases in proportion to the product of the length times the diameter of the plasma tube. This suggests that long tubes are more powerful than short ones. Again there is a catch. Short tubes operate readily at 6,328 angstroms, whereas those a meter or more in length tend to function in the infrared region instead of in the visible part of the spectrum. I am told that long tubes can be forced to work in the visible range by the strategic placement of magnets along the tube, but I have not tried the experiment. Tubes ranging from 15 to 40 centimeters in length appear to work quite adequately in the visible region. Mine measures 34 centimeters.

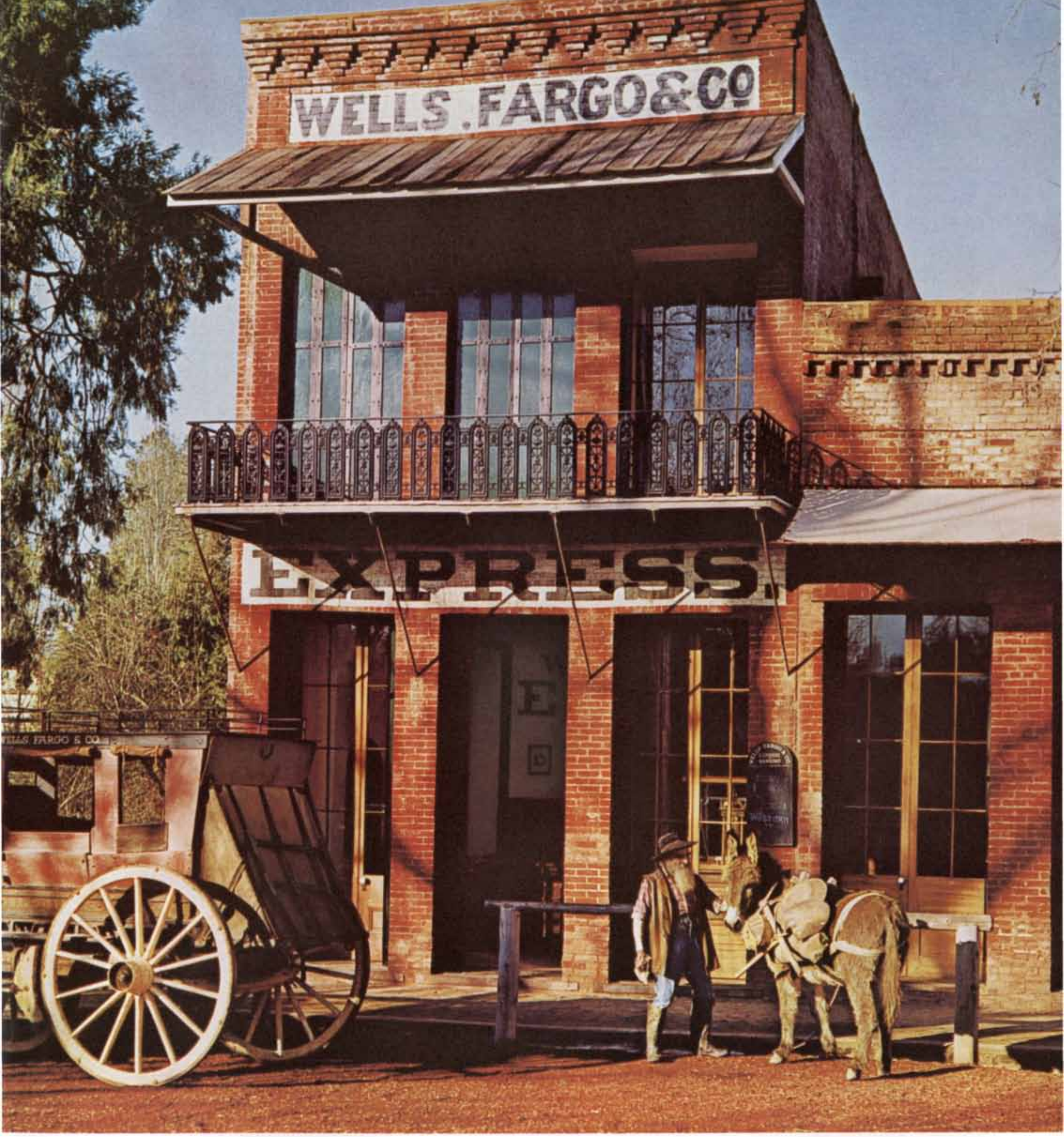
The mirror system functions somewhat like a resonator. It can consist of various combinations of spherical and flat mirrors. A system that is easy to adjust and to maintain in adjustment employs a pair of facing spherical mirrors separated by slightly less than one radius. My mirrors were figured to a radius of 60 centimeters. The spherical surfaces have a separation of 57.5 centimeters. The mirrors are mounted in easily adjustable cells supported by fixtures that can be removed and returned to the base without disturbing the alignment of the mirrors with respect to the axis of the tube. All essential dimensions of the laser's hardware, of the light projector used for aligning the mirrors and of the miter box for sawing the envelope at the Brewster angle are specified in the accompanying illustrations. Alternate design schemes are possible and perhaps desirable. The reader is encouraged to improvise.

Gas lasers can be energized by alternating current of either high or low frequency and by direct current. The direct-current types that employ heated electrodes have a long service life. Making the heated electrodes, however, is an intricate job that I am reluctant to undertake now. My laser is equipped with cold electrodes made of aluminum in the form of small cylinders and is energized by a conventional neon-sign transformer of the constant-current type. The primary voltage is controlled by a Variac, a variable-voltage transformer. When power is applied to the primary of the neon-sign transformer, the secondary winding maintains 18 milliamperes through the load at a maximum potential of 9,000 volts. Experiment demonstrated that the laser beam reaches maximum intensity when 85 volts is applied to the primary of the neon-sign transformer.

I began building the laser by making the getter-electrode assemblies. The getter is a small metallic trough, filled with barium, that is connected at the ends to a loop of wire. When it is installed inside an evacuated glass envelope, the loop can be coupled to a high-frequency electromagnetic field and heated by induced current to vaporize the barium. That is the conventional procedure. I do not own an induction-heating apparatus of this type, so I cut the loops of two getters, spliced the pair in series by means of a pigtail joint and hooked a pair of Dumet leads to the free ends as shown in the accompanying illustration [*top of this page*]. Dumet is a spe-



Miter box for sawing glass at the Brewster angle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But the photographer who must wait to see his pictures doesn't have the advantage of

First try, too far away and not very well composed. And so impersonal it could be anybody's family.

that continuous experience. He's practicing a stop-and-go art form. When the average person takes a picture he doesn't know how it turns out 'til days later. Even if he's interested in profiting by his mistakes, it's too late. The location and lighting are different. The mood is gone. The creative continuity is shattered. (Even getting involved in processing your own pictures doesn't do much except shorten the

A minute later, getting better but still not perfect. Now get a little lower and wait for the right moment...

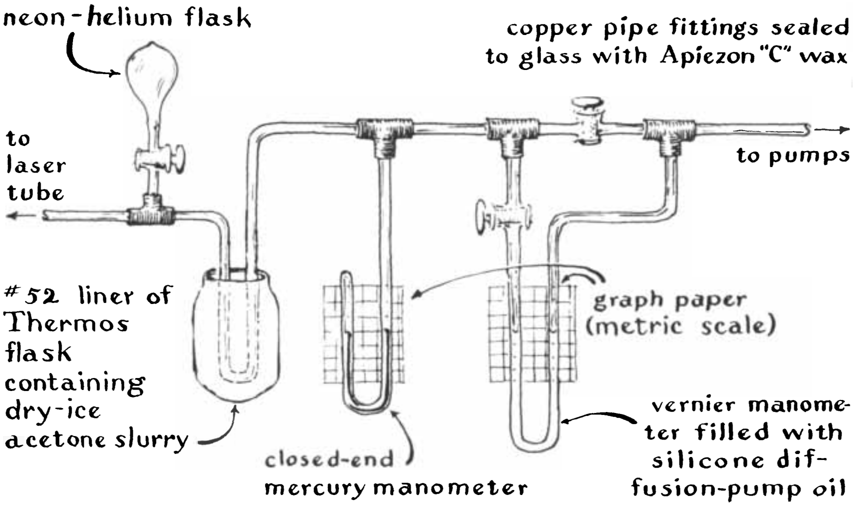
waiting period.) Small wonder there aren't more "Sunday" photographers.

The one exception to all this: a Polaroid Land camera. We think it's the key to an enormously satisfying way to express yourself.

If you take black and white pictures, you see them in 10 seconds. If you prefer color, one minute is all it takes. And because you see your results so fast, you soon learn the aesthetic values of fine photography. Interpretation, composition, lighting and total effect are all apparent, and changeable, on the spot. The challenge and the satisfaction become a little sharper with every picture you take.

Whether you're a camera bug or not, we suggest that you borrow a Polaroid Land camera and try it out. We can almost guarantee that your second picture will be better than your first. And that your third will be better than your second.

And that's what Polaroid Land photography is all about.



Schematic arrangement of system for metering gas

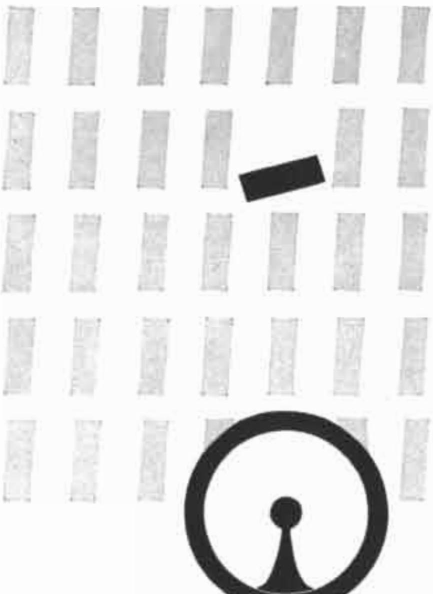
cial alloy wire that seals readily to soft glass. I fire the getters by hooking the Dumet leads to a six-volt transformer that is energized through the Variac. The assembly becomes red-hot when a potential of about two volts is applied to the leads and yellow-hot at three volts. At that temperature the barium vaporizes and condenses in the form of a dime-sized black film on the glass surface of the evacuated tube. The pair of getters draws seven amperes at three volts. I apply the heating current slowly, allow about a minute for the temperature to rise to yellow heat and then switch off the power promptly when the film of condensed barium becomes almost opaque. The units contain enough barium for about five such firings. It is easy to make the mistake of increasing the power too quickly and exploding the wire.

The pigtail splice between the two getters serves as the support for the cylindrical electrode: a ribbon of clean aluminum foil in the form of a single, slightly overlapped turn. I make it by winding the foil around the end of a six-millimeter glass rod. One end of the resulting cylinder is then twisted around the getter pigtail. Before touching the getters and foil thoroughly clean your hands, as well as any tools, with carbon tetrachloride. Take the foil from the inside of a new roll.

The getter-electrode assemblies are installed in two short lengths of glass tubing that become side arms of the glass envelope. One side arm is equipped with a smaller tube for exhausting and backfilling the envelope. I then flare the ends of the plasma tube alternately by blowing a small bulb on one end, exploding the softened bulb and shrinking its circular edge in the fire until, by trial and error, the flared end fits the inside of the envelope to within a half-millimeter, or closer if possible. A dimple is next sucked in the envelope as a stop for the plasma tube. The outside of the plasma tube is cleaned with fuming nitric acid, rinsed with distilled water, dried in the flame and slipped into the envelope against the dimple stop. The second dimple is then made in the envelope to secure the tube loosely. Next the side arms containing the getter-electrode assemblies are joined to the envelope. Incidentally, if you have the glasswork made commercially, you will miss a lot of fun. Working with hot glass, particularly in the case of simple apparatus such as this, is not nearly so difficult as is commonly supposed. I discussed glassblowing in this department in May.

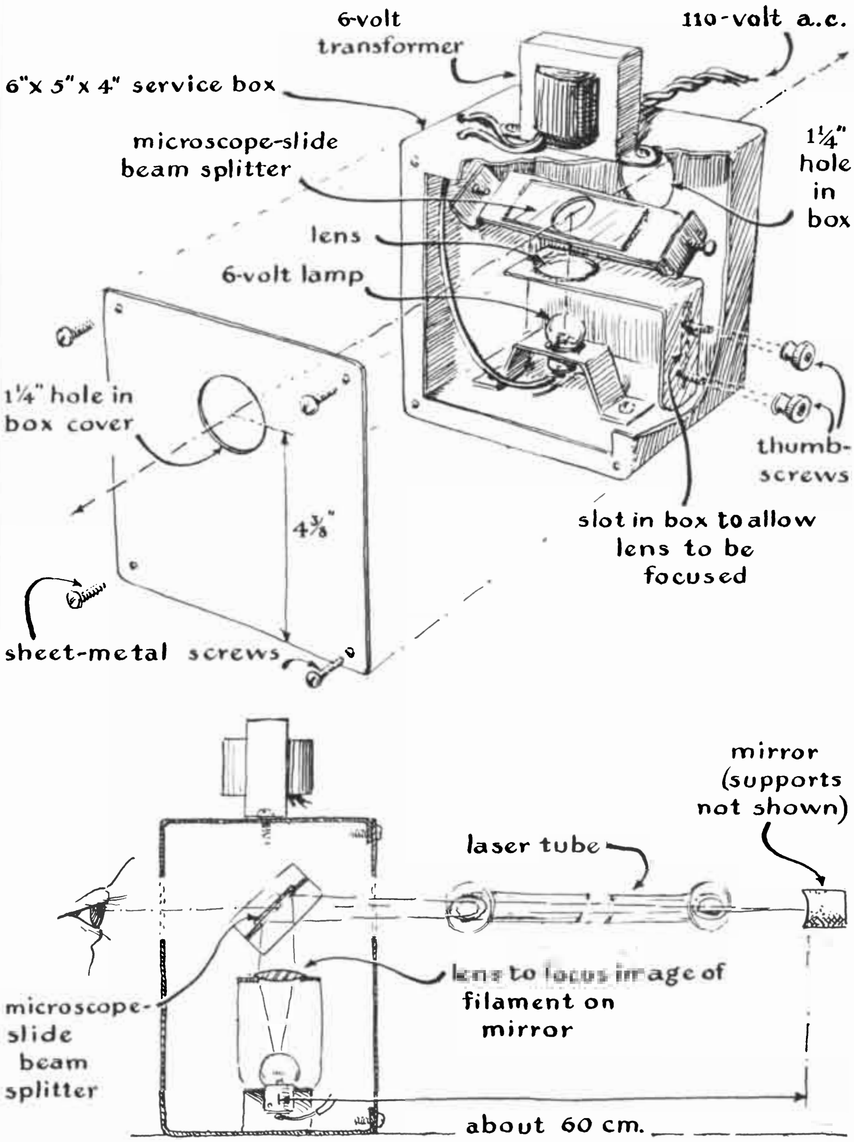
The glass construction is completed by sawing the ends of the tube at the Brewster angle. If you have access to a diamond saw equipped with an accurate fence, the job will take about three minutes. If not, build the miter box shown in the bottom illustration on page 232. With this device the cuts will require about 10 minutes each. Keep plenty of abrasive slurry on the copper blade, let the weight of the saw do the work, use about 60 strokes a minute and take it easy as the saw cuts through the glass. Before making the cuts, plug the ends of the tube with wads of clean absorbent cotton and cement the glass to the right-angled aluminum holder.

The cut ends must be lapped to make a vacuum-tight fit with the Brewster windows. Begin the lapping operation with approximately 400-mesh grit, either



Alundum or Carborundum, using a two-inch square of quarter-inch plate glass as the tool. When the pits that were made in the glass by the saw have *almost* disappeared, shift to 600-mesh grit and continue lapping until pits left by the 400-mesh grit have *almost* disappeared. Examine the work through a magnifying glass. The size and number of the remaining pits indicate where the glass is being removed and by how much. If pits disappear slower at one point than at another, exert more grinding pressure on the plate glass above that region. The idea is to remove glass evenly over the entire area of the cut end, thus preserving the Brewster angle. The job requires about 20 minutes. When the lapping is finished, remove the tube from the fixture by warming the pitch that serves as cement. Take out the cotton wads. Clean the soiled inner ends of the envelope with a swab of cotton moistened first with acetone and then with distilled water. Dry the cleaned ends in a gas flame. Remove the pitch with turpentine.

To install the Brewster windows clamp the tube in an apparatus stand or a comparable fixture, connect the side-arm tubing to the mechanical pump of the vacuum system and start the pump. Simultaneously place the cleaned Brewster windows flat against the cut ends. Suction will hold them in place. Using a toothpick as a spatula, apply a thin layer of Torr Seal epoxy cement to the exterior of the joint between the windows and the tube ends. (Torr Seal is manufactured by Varian Associates,



Light projector for aligning the laser mirrors

Palo Alto, Calif.) Watch the inner surface of the windows carefully for any trace of vapor from the cement that may be sucked into the tube. If the least trace of vapor appears, the lapping job is defective and must be corrected. No cement should seep through the joint. The curing time of the cement can be shortened by applying heat to the joints by means of a pair of 150-watt incandescent lamps placed about 15 centimeters away. Recommended curing temperatures are specified on the package by the manufacturer. When the joints have cured, apply a coat of Dow Corning 806A silicone resin over the cement, all glass joints and the Dumet seals.

Next fire up your vacuum system and connect the tube. The system should be equipped with at least two manometers: one a closed-end mercury type and the other a conventional manometer half-filled with high-grade oil of the kind used in a diffusion vacuum pump. Scales for the manometers can be made of graph paper calibrated in millimeters [*see illustration on page 237*]. The oil manometer acts as a vernier gauge, the pressure readings in millimeters being converted to torrs by dividing the specific gravity of mercury (approximately 13.5) by the specific gravity of the oil and then dividing the difference in the height of the oil in the two arms of the manometer, in millimeters, by the quotient. For example, if the ratio of the specific gravity of the mercury to that of the oil is 16, a manometer reading of 32 millimeters—the difference in the height of the oil in the two arms—indicates a pressure of 32/16, or 2 torrs. You can find the specific gravity of the oil with sufficient accuracy by weighing 10 milliliters and dividing the weight (in grams) by 10.

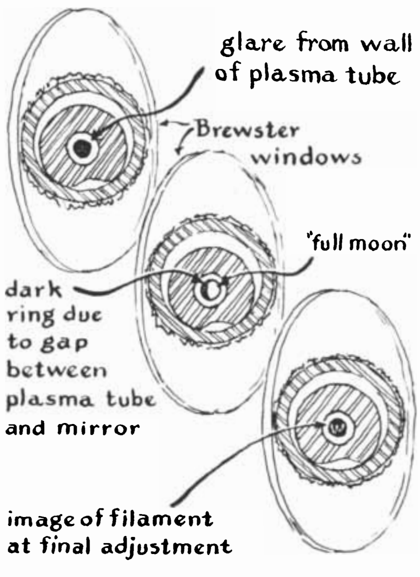
Your vacuum system should also contain a McLeod gauge, primarily for ensuring that the system pumps to the required 10^{-5} torr. If you do not own a vacuum system, you may be interested in building the inexpensive one described in the *Scientific American Book of Projects for the Amateur Scientist* (published by Simon and Schuster in 1960).

My system uses a two-stage mercury-jet pump. For valves the system has inexpensive glass stopcocks. I lapped them with 600-mesh grit to make a vacuum-tight fit and lubricated the stoppers with a thin film of high-vacuum grease. One stopper was modified to function as a leak valve for admitting the required minute volume of helium-neon gas to the laser tube. Deep scratches about six millimeters long that tapered upward to the surface were cut counterclockwise from each end of the hole in the stopper. The tips of the scratches conduct gas at a convenient rate when they engage the openings of the barrel.

You will receive the helium-neon gas in an all-glass container. The outlet tube contains an easily broken seal. To tap the supply clamp the flask temporarily to a support, with the tube pointing downward. Insert a loosely fitting glass marble or a short length of heavy glass rod in the tube, join the leak valve to the tube with Apiezon "C" wax and connect the valve to the vacuum system by 50 centimeters or so of copper capillary of the type used in the thermostats of gas stoves. Open the leak valve fully and exhaust the tube to 10^{-5} torr, or lower if possible. Then close the valve and invert the flask quickly so that the marble falls and breaks the tip of the seal.

The laser tube is next exhausted, cleaned by discharge bombardment, filled to the required pressure and sealed off. For support the tube can be assembled to the base, in which case the laser can be tested before the seal-off.

Pump the tube down to 10^{-5} torr or lower, then backfill with helium-neon to a pressure of approximately 5 torrs. Connect and switch on the 9,000-volt transformer. The tube will fill with colored plasma—reds, greens, blues—and may get quite hot, on the order of 100 degrees centigrade or more. After five minutes switch off the power, repump the tube and repeat the cycle. Continue the procedure until the reddish color predominates, tinged only slightly by blue. This may require four but probably not more than six cycles. Before pumping down at the end of the last bombardment heat the getters to dull red. The tube should be positioned so that both getter assemblies are suspended vertically by their leads, otherwise they may sag enough to touch the glass when heat softens the metal. After about three minutes advance the Variac gradually to fire the getters. Watch the glass wall adjacent to the getter assembly as the temperature of the metal approaches yellow heat. A sooty film will appear. As the film reaches opaqueness switch off the power. Having pumped down, refill the tube to a pressure of *not more* than 2.7 torrs (assuming a plasma tube of two-millimeter bore). Energize the tube again. The plasma should now appear solid reddish-orange. When placed between a pair of properly adjusted mirrors, the tube will now function.



Images during mirror alignment

To adjust the mirrors, first remove either of the cell fixtures from the base and replace it with the light projector. Adjust the lens of the projector until the image of the filament is in focus on the front surface of the distant mirror. Then, using your hand as a screen at the near end of the base, locate the beam reflected by the mirror and adjust the cell to center the beam on the distant end of the plasma tube. Look through the projector's beam splitter (the microscope slide) into the bore of the plasma tube; in the center of the glare that is reflected by the inner wall you may observe a minute disk of light that resembles the full moon surrounded by a thin, dark ring. The "moon" is the reflection of scattered light from the projector. Adjust the cell in any direction that causes the disk to brighten. Ultimately it will become dazzling as the reflected image of the filament comes into view [*see illustration at left*]. This completes the adjustment of the first mirror. Remove the adjusted cell fixture carefully and similarly adjust the second one. When both adjusted mirrors have been assembled to the base, connect the neon-sign transformer to the tube and switch on the power. Usually it is now necessary to "fiddle with the screws." Just rock the adjustment screws back and forth a degree or so, one after the other. Suddenly the beam will appear—you have a laser! Seal off the tube.

The beam, when it is directed against a screen, will doubtless appear in the form of a symmetrical pattern of dazzling red spots—perhaps only a pair, maybe a rosette of eight spots or some other geometric design. The pattern can be greatly altered and perhaps intensified simply by fiddling with the adjustment screws. Each pattern of spots results from a unique set of paths taken by rays that oscillate between the mirrors. The various patterns of vibration are known as "modes." Observe too the scintillating granularity of the light. According to E. I. Gordon, a physicist at the Bell Telephone Laboratories, this striking effect actually arises in the eye or any other image-forming device, such as a camera. Each dazzling speck, he explains, marks a point of constructive interference between coherent diffraction patterns; the size of the point is determined by the diameter of the pupil. Phenomena such as the granularity suggest many fascinating experiments that can be made with laser light. Some will be discussed from time to time in this department.

Now, a word of warning: The laser is a hazardous apparatus. *Never look directly into the beam.* Coherent light of this intensity can damage the retina permanently and may even cause blindness. When other people are in the room, block off all beams at points close to the apparatus: the two beams from the ends and the remaining four of lesser intensity that come off the Brewster windows. The 9,000-volt output of the neon-sign transformer is lethal. Insulate the full length of the leads with abutting pieces of glass tubing. Never touch the terminals of the laser when the transformer is plugged into the power line even if you are certain that the switch is off.

Parts and materials for constructing the laser can be procured from the following suppliers:

Perkin-Elmer Corporation, Electro-Optical Division, Norwalk, Conn. To encourage student experimentation in optics this firm has developed special dielectric mirrors of adequate quality for apparatus of the type described in this article.

Edmund Scientific Co., Barrington, N.J. This organization stocks lenses and related materials.

Morris and Lee, 1685 Elmwood Avenue, Buffalo, N.Y. 14207. Air pumps, pressure gauges, valves and accessories for vacuum systems are made specially for amateurs by this organization.

Henry Prescott, Main Street, Northfield, Mass. This supplier specializes in all materials required for constructing and experimenting with the laser, including dielectric mirrors, vacuum systems, helium-neon gas, glass components, getters and related essentials.

BOOKS

Experiments toward a humor of science

by James R. Newman

PSYCHOLOGY IN THE WRY, edited by Robert A. Baker. D. Van Nostrand Company, Inc. ($1.75). A STRESS ANALYSIS OF A STRAPLESS EVENING GOWN, edited by Robert A. Baker. Prentice-Hall, Inc. ($3.95). THE SCIENTIST SPECULATES, edited by I. J. Good. Basic Books, Inc., Publishers ($6.95).

Is science humor possible? There is no obvious reason why it should be; the substance of science, ranging from acoustics to zymurgy, is scarcely designed to make laughter hold both its sides. To this generalization, however, there are some footnotes. Stephen Leacock was a man who could make anything funny by gentle ridicule, and he produced delicious pieces about such lofty and seemingly invulnerable subjects as cosmology. Moreover, it is true that science, like any other profession, has its share of inside jokes. There are such celebrated tales as the one about Wolfgang Pauli, who, it was said, was so completely the theoretician that if he even set foot in a laboratory some expensive piece of apparatus shattered spontaneously.

Finally, of course, there is a body of literature that punctures the solemnity, the pretentiousness and the phony weightiness of many a report in sociology or psychology or urban planning or anthropology. Science making fun of itself is heartening and healthy. There can scarcely be enough of such satire; there is in fact much too little. Although the social sciences are both vulnerable to it and their language in large part is accessible to the average educated man's understanding, so that the satirist can make fatuities perfectly plain, the formidable technical language of physics or chemistry or mathematics is a shield against attack. Anyone can tell when the emperor is naked, but if he is encased in a grand suit of armor, who can tell if he is a puny shrimp or a real man?

The three books reviewed here may be regarded as experiments toward a humor of science. Two are meant exclusively for laughs; the third, although it has another intent, is more genuinely humorous.

Psychology in the Wry is mainly an "in" book. It consists of 20 selections by psychologists and scholars that satirize some of the characteristic idiocies of the study of human behavior. Edgar F. Borgatta considers the traumatic effects of "deumbilification" and the importance of "mammary envy." We are told that in many societies a cow is more valuable than a wife for this very reason. All girls can at least hope to have a glorious bosom and thus to become interchangeable with cows; males hope in vain. Norman R. F. Maier formulates a law cognate to Parkinson's: If facts do not conform to a theory, they must be disposed of.

A disease known as petty paranoia that afflicts clerks and officials is described by Craig M. Mooney. An Associated Press dispatch from Columbus, Ohio, reads as follows:

> A Manchester decorator hurled a chair through a window of the Canadian Government's citizenship and immigration offices in London Saturday after a ship sailed for Canada without him.
>
> "I did this because it seemed the only way I could draw attention to this case," Albert Hartwell, 41, said in court.
>
> He said he had applied to emigrate to Canada, filled in the necessary forms, had a medical examination, was accepted, and paid $168 for passage on a ship sailing Saturday.
>
> Five days ago he got a letter saying his wife had to be medically examined.
>
> He said this must have been a mistake as his wife was not travelling with him. He wrote back noting the mistake but received a telegram telling him to bring his wife from Manchester to London. On Friday he was told at the immigration office he could not sail because his wife had not been examined.
>
> The shipping office said he would have to apply for later passage. He went back to the immigration office and explained that having sold his home and left his job he had only enough British money to get through Saturday.
>
> The magistrate said Hartwell seemed to have broken the window in a fit of temper.
>
> He was freed on a charge of willful damage but ordered to pay two pounds, the cost of replacing the window.

Even more touching is a Reuters dispatch from London:

> Since eating a gumdrop, 11 year old Charlotte Sexton hasn't been to Westerville Hawthorne School's Sixth Grade for four days. It seems Charlotte defied a regulation banning eating on buses. Until her parents reassure the driver she will eat no more, Charlotte is not allowed to board the bus. Ray Morris, Westerville School Superintendent, said the regulation has been in effect at least 17 years. Its purpose is to prevent children from throwing orange peels, apple cores and candy wrappers at each other on the bus, stirring up trouble.

There is an article on the anal eroticism of the military and an imaginary review of *In the Bowels of the Soul* by one Wilhelm Oenkschwein Ritter von Kluernk. This book holds that the basic human urge is a deep instinct to return to the saltwater seas from which our ancestors emerged: "Man makes money to return to the sea (symbolized by Miami Beach) or is overcome by the... inhibitions of our rotten, mechanistic, sanitary society and moves inland in a masochistic reaction formation (symbolized by Kansas)."

Such stuff may be mildly amusing, but the reader will obviously be able to contain his risibility. Baker's other collection earns about the same grade. Its title essay—"A Stress Analysis of a Strapless Evening Gown," by Charles E. Siem—comes close to being very funny, with deadpan engineering chatter about tangential forces, coefficients of friction, safety factors, bending moments, vectors and the other matters that have to be considered. Also entertaining is a systems-analysis approach to the working of swivel postboxes. F. E. Warburton offers a brief natural history of the tern. Terns, we are told, "don't stick around in the winter. Arctic terns, for example, spend the summer at the North Pole. When it begins to get cold, they fly south to spend the winter at the South Pole, where it is summer. Having spent the winter at the South Pole, they fly north to avoid the winter, arriving at the North Pole in the summer while it is winter where they wintered. . . . Terns are found in pairs, if they are good, because one good tern deserves another."

Among other inviting targets of essays in the book are the lore of the Abominable Snowman, U.S. missile programs, research administration, computer literature, anthropological studies of remote tribes, cybernetics, the history of science, the applications of mathematics to almost anything. I was taken by Joel Cohen's "The Pejorative Calculus," in which one can prove that all horses are the same color, every horse has an infinite number of legs, Alexander the Great did not exist and he too had an infinite number of limbs.

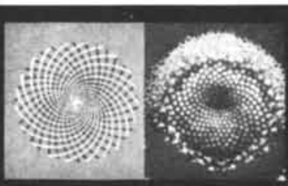
The Scientist Speculates has largely to do with partly baked ideas. We all evolve P.B.I.'s while awake. (Dreams are fully baked: they are what they are, untrammeled by space, time and causality, not susceptible of improvement or validation.) It has been suggested that most scientists carry on an almost endless "internal monologue." Usually it consists of questions. The biggest question is what question to ask. Man has long comforted himself with the thought that if the question could be asked, it could be answered. Some of the best ideas are nothing but questions. The kind of science one gets—like the kind of ethics or politics—depends on the kind of questions one asks. It is easy to imagine a different physics or biology or even a different mathematics if the questions had been different.

Good's collection is full of questions. The entries take many forms, from little jokes and paradoxes to elaborate speculations and theories. One of the merits of the anthology is that it draws no clear line between the scientist flipping around irresponsibly like a dolphin in the sun and the scientist soberly venturing hypotheses yet to be tested. Almost all the partly baked ideas are presented seriously and almost all have their innocent, funny side. The contributors were invited to play and many caught the spirit. J. D. Bernal said that he had been producing crazy ideas for so long he couldn't really do much damage to his reputation by going on; he recollected his son's reply a couple of years ago "when I suggested he might add his name to a paper of mine in which he had helped considerably. He said: 'It is all right for you but I have my reputation to consider.' "

Among the many diversions in the book is R. H. Thouless' examination of noncommunicating discourse, a common feature of the use of language. There are, for example, the vacuous affirmations: "Ladies and Gentlemen. We have gathered together in this great hall. We have come from far and near. A year has passed since our last coming together, a year of hopes and disappointments, of joys and sorrows. . . ." These communicate nothing because they state nothing that could conceivably be unknown to those on whose ears they fall. Thouless offers illustrations from standard writings in psychology: "The young child learns to use the word 'Mama' to refer to his mother"; "The student whose mind is wholly engrossed in a mathematical problem is not aware of what is happening around him"; "Men act upon the world, and change it, and are changed in turn by the consequences of their action. . . ." He envisions a moderate program for the suppression of N.C.D. "I hope," he concludes, "that a sufficient number of people would agree with me that a world in which a speaker sat down when he found he had nothing more to say would be a better world than one in which he went on talking."

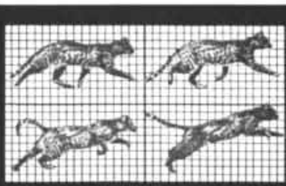
Another selection is a technical glossary that incorporates material from two other articles. Thus:

"It has been long known that . . ."	=	I haven't bothered to look up the original reference.
"While it has not been possible to provide definite answers to these questions . . ."	=	The experiments didn't work out, but I figured I could at least get publication out of it.
". . . accidentally strained during mounting."	=	. . . dropped on the floor.
". . . handled with extreme care throughout the experiments."	=	. . . not dropped on the floor.
"It is clear that much additional work will be required before a complete understanding . . ."	=	I don't understand it.



FROM THE MYSTERIOUS GEOMETRY OF NATURE . . .

The daisy head bears curious relationship with the mathematical Fibonacci series, produced by starting with 1 and adding the last two numbers to arrive at the next: 1, 1, 2, 3, 5, 8, 13, 21, 34, etc. Daisy head has 21 clockwise spirals, 34 counter-clockwise!



. . . TO THE CALCULUS IN A STARTLED CAT . . .

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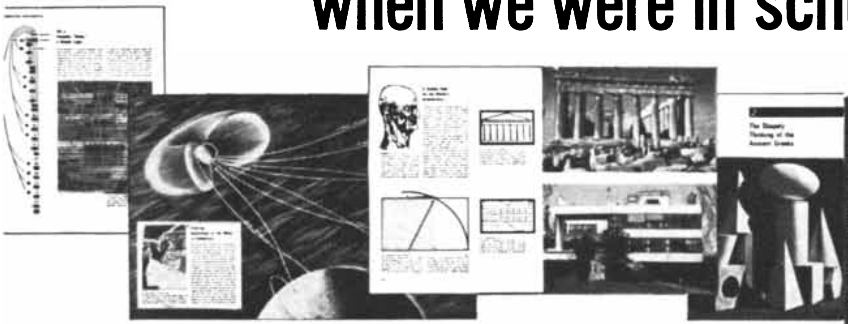

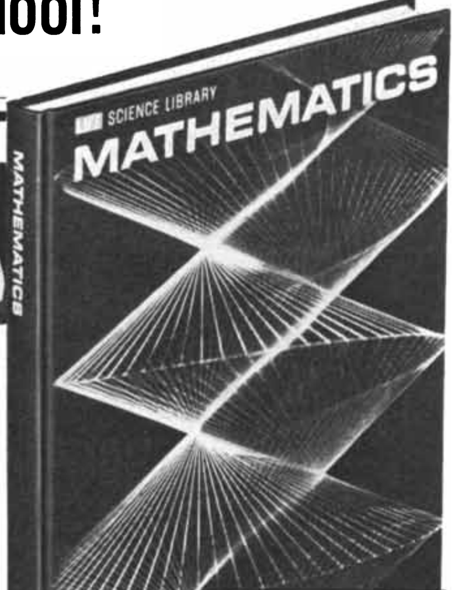

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"Typical results are shown."	=	The best results are shown.
"It is generally believed that..."	=	A couple of other guys think so too.

Christopher S. O'D. Scott inspects the question of whether or not anomalies are normal. Most of us realize that impossible things are constantly happening. Take the wayward antics of everyday objects. I put my glasses on the desk and two minutes later they are not there. No matter how hard I search I can't find them. They turn up the next day on the mantelpiece. I know I never went near the mantelpiece, but what can I say? Similar mysteries occur in the laboratory. Experiments go wrong, and when they do, "we say the materials were impure, or make some equally *ad hoc* excuse. We then repeat the experiment until it goes right; we certainly don't take the anomalous result seriously as a new natural phenomenon." Perhaps we should; perhaps we are obliged to consider whether J. B. Rhine's results are normal anomalies to be explained as the disappearing glasses might have been explained. Also in the field of extrasensory perception, division of precognition, is the suggestion by the late J. H. C. Whitehead that we may each have a guardian angel out in space, to whom we send out thought waves faster than light and who then reflects them back. According to the special theory of relativity the signal would return before it was sent, which might explain precognition.

Does large-scale public wishing affect the weather? William E. Cox says that we might obtain evidence on the question by examining weather records and determining, for example, whether there is more sunshine or less rainfall on days when fine weather is particularly desired, such as on Sundays, than on other days. According to newspaper records there were 211 days when the sun did not appear by afternoon press time in St. Petersburg, Fla., from 1910 through 1957. Only 11 of those days were Sundays. In the absence of parapsychological forces the odds of there being that few cloudy Sundays are 20,000 to one.

Good himself puts forward a theory that has the unusual feature of being unbelievable if it is true. The theory is that we are on earth as a punishment for crimes committed in heaven; we have to serve our term according to the magnitude of our heavenly offense. "Even if we are happy here, it is a poor sort of happiness in comparison to what we shall have when we return. But part of the punishment is that we cannot believe the theory. For if we did, the punishment would not be effective."

In the field of sociology Colin Cherry frames a commendable hypothesis concerning highway accidents. It has to do with the fact that the driver of an automobile is held incommunicado by the very nature of driving. This is an almost intolerable constraint, particularly when the constraints of various codes and regulations are added to it. You are driving along when you notice in your rearview mirror "another car closing up behind you, jockeying for position and settling down two feet behind your rear-lights. It takes every ounce of self-control to accept this situation, and to feel well-disposed, as you sense on your neck the hot breath of this monstrous hate-object." With the situation reversed, the same sense of irritation and frustration arises as you try to pass some middle-of-the-road hog. Now, says Cherry, think of how these hatreds could be dispelled by a few courteous words. (Well, anyway, a few words.) Without such communication all you can do is honk or try to get out and hit the other driver; you can express none of the finer shades of emotion. Cherry's recommendation is that cars should be fitted with miniature transmitter-receivers with a range of about 100 yards. He is convinced this would virtually eliminate collisions.

"Discovery," Albert Szent-Györgyi once remarked, "consists of seeing what everybody has seen and thinking what nobody has thought." Any number of speculations in this anthology fall under this epigram, even if they are only partly baked and may never enjoy, let alone survive, the full test of the oven. A nice example is Bernal's proposal for meeting the universal need for water.

There are many ways of getting fresh water for crops, drinking or other purposes, from the most primitive arrangements to atomic-energy desalting schemes. Bernal's plan, which he has been thinking about on and off for some 30 years, is "extremely simple and almost automatic." It is based on the obvious fact that water vapor is just half as heavy as ordinary air and therefore will rise of itself. The common impression that water vapor goes up from the sea because it is heated is wrong; it goes up because it is light. The object

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is to catch water vapor, which is salt-free, as it rises from the sea, conduct it to some place where cooler air will condense it and then run off the product to where it is needed. The key apparatus is a chimney, which is to be filled with steam. Bernal asks us to imagine a desert coastline with a cliff or bluff rising to a hill plateau perhaps 3,000 feet high. Such configurations are found on many desert coasts of the world: in Arabia, North Africa and Australia. In front lies a beach or marshy coastal plain and behind there is a desert. "All that is needed here is to build an arched tunnel for the steam all the way up the side of the hill. Water from the sea, superheated by the sun, is allowed to flow into pits where it meets a stream of preheated dry air and evaporates, leaving the salt behind and turning into vapour which can be further heated by the sun. The mere existence of the chimney will ensure that a steady stream of water vapour will flow right up to the top. There, all that is necessary is to reverse the process and condense the vapour to water again." The sequence of events is not unlike the evaporation of water from the sea and its condensation as rain on high coastal mountains. Here one directs the water vapor and does not just let it go loose. "The whole process would be completely automatic, would require no power and no attention. The sea-water would just run in at the bottom and the fresh water would run out at the top."

Obviously such a scheme presents a large number of difficulties, some purely technical, some economic. Bernal believes they can be overcome. For instance, the channels for the vapor have to be very big and might be enormously expensive. Perhaps thin plastic sheeting could be used to cover a light framework, providing a series of channels dozens of yards across. No doubt this would be an expensive means of getting water compared with nature's way of picking up the vapor and dropping it somewhere else as rain; but if nature were so obliging everywhere, there would be no shortage of water. At any rate, an enterprise such as this would be suitable where it paid most to get water, that is, in desert regions near the sea. In such regions, Bernal points out, there is not so much need for subsistence agriculture as there is for vegetables and other cash crops.

Wordsworth described a philosopher —by which he meant a natural philosopher, not a metaphysician—as "one that would peep and botanize upon his mother's grave." Scientists will dare almost anything to find out. That is how we come to such ingenious concepts as the cobalt bomb. There are other dangers. For instance, if you seriously entertain the notion of making people immortal, you threaten to make the entire globe as crowded as Coney Island and life itself such a bore that man will yearn for the happy days when it was merely nasty, brutish and short. James S. Hayes's scheme for immortality avoids certain of the obvious disadvantages. It depends to some extent on the large-scale use of telepathy and the design of a high-capacity channel for the transmission of information from one brain to another.

Consider the following situation. "Your body, by now perhaps 130 years old, and consisting for the most part of prosthetic devices, is lying partly in a hospital bed and partly on the surrounding shelves and tables. It will soon become impracticable to maintain the necessary oxidative reactions in your brain. Someone younger is wheeled in, his own mental activities are temporarily suspended and the process of transferring to his brain the whole of the information content of your own begins. As it nears completion, you begin to be aware of sensations deriving from his exteroceptors and proprioceptors as well as your own. You see with his eyes, hear with his ears. A sympathetic technician asks what was formerly the other fellow's body how you are feeling, and on receiving a satisfactory reply turns off what was, a few minutes ago, your heart. After a short rest period, the temporary block on access to the experiences acquired by your new body on its own account is removed, and you begin to remember having been him as well as yourself. But this in no way diminishes you, nor decreases your certain knowledge of your own existence. The process must clearly be symmetrical: to your host he is still himself with an extra lifetime of experience to draw upon."

One can think of personal identity as consisting of an ordered store of information. If this information can be transferred *in toto* to some other chap, the donor can be said to have been preserved even though his chassis is abandoned and "dies." Of course, one does not have to transfer everything; in fact, this would be difficult. Moreover, one can imagine transferring the same parts to many people, so that one personal identity could simultaneously be made available to several brains and simultaneously inhabit many bodies. "A time may come when almost every person

one meets will have at least some part of his identity in common with oneself. Under these conditions the death of a body would not be a reason for grief, as now death is a reason for grief."

The preservation of personal identity is circumscribed by certain theoretical considerations. Quantum-mechanical principles set an upper boundary to the information-storage capacity of the human brain; hence there is "an impassable limit to the number of 'life-experience units' which could be made available in their entirety to any single human body." There is at least one safeguard against the absolute tedium of immortality. We cannot store up in one little head all the ramifications of the tensor calculus and how Aunt Gussie felt about the stays in her corset every time she sat down. Another possibility—the perpetuation of life experience in the memory banks of giant computers—will not suffice to preserve every sigh; at some point "even a solid sphere of organized matter expanding at the velocity of light will be inadequate to contain us all." The properties of the universe will therefore reinstate an equivalent of death. We will be forced to toss out of the great attic of memory some of the toys we judge to be no longer useful. Death will not lose out entirely, although it will "have lost its personal sting."

Three other selections deserve mention; indeed, they are entitled to a thorough discussion, but I can mention them only in passing. Harlow Shapley conjectures about Lilliputian stars that may bear life. Eugene P. Wigner links the mind-body question to quantum mechanics and argues for the effect of consciousness, in a non-ESP sense, on physical phenomena. David Bohm proposes a topological formulation of the quantum theory.

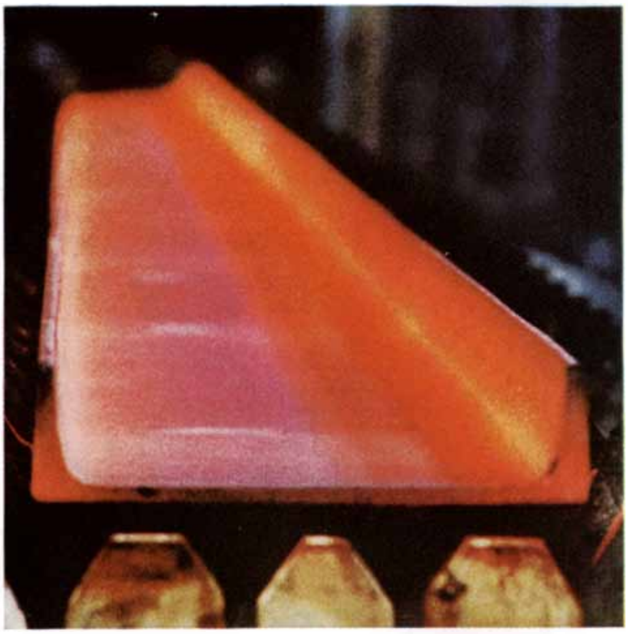
There is one final point in favor of Good's collection. Not a single essay in it relates to astronautics. Perhaps this is an indication that the frontiers of the imagination will someday not be confused with those of space.

Short Reviews

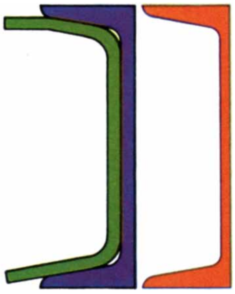
THE ROOTS OF EVIL: A SOCIAL HISTORY OF CRIME AND PUNISHMENT, by Christopher Hibbert. Little, Brown and Company ($6.95). Can a gregarious, aggressive species such as man develop a civilization? Yes, if the advantage is great enough and the intellectual prerequisites are available. Man comes to love his ingroup and that piece of altruism in him forms a base for an evolving civilization. But man also tends to fear and thus to hate the strange, the outgroup—not only his certain enemies but also all those who thwart him: the criminals, the heretics and the insane. Nor does he readily extend his altruism to cover barbarians and remote peoples. This well-documented, fact-packed book traces for 1,000 years the history of man's response to those who offend his will and who today, when civil progress has at last arrived, break his laws. The volume tells the history of man's heartless cruelty to this outgroup against which his hatred is so readily evinced and which, only slowly and with difficulty and not yet completely, he brings within the scope of his understanding and compassion. "Reform," as the spread of this understanding of the offending minority is called, is hardly more than a matter of the past two centuries within the millennium under consideration, and the shift of punishment away from cruelty and mayhem, of justice away from venality and of the law away from retribution that does not deter has been slow. It has today at its best scarcely more than reached the point at which society forgets about the criminal, putting him away in a crowded crime school where he learns to hate the social system and where the young in misdeed are taught by habituated malefactors the ways by which to succeed at the expense of the enemy, society.

Hibbert writes well, and half as many facts in his hands would be convincing. In England up until the 15th century one was fined for aggression—100 shillings for a murder, 50 shillings for an eye gouged out but only sixpence for the removal of a toenail. Thereafter punishment became more severe. With heresy added to treason as a crime, torture became a preliminary to hanging, drawing and quartering, with the head, preserved by parboiling, stuck on a spike on London Bridge. People in the pillory, with feet not quite on the ground, were degraded by having offal thrown at them and were occasionally killed by stoning. After a hanging the crowd sometimes fought for the body of the deceased. The finger bone of a murderer in one's purse was supposed to keep it from ever being empty, and it could not be empty with such a precious relic always unspent. Reform, as it inched its way in, concerned the law, the police and the prisons. When the court pronounced a man innocent, he still had to pay the jailer for his keep before he could be let out. Maybe he had spent weeks chained spread-eagled to the damp, rat-infested floor, with heavy iron

Special Report: How U.S. Steel put 3 lbs. of steel into a 1-lb. package

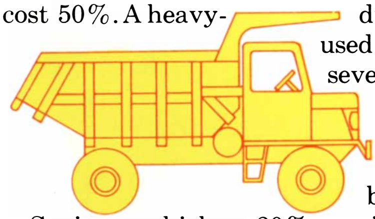


Long ago it was discovered that the strength of steel could be tripled by adding alloys and heat treating. But not until recently have heat-treated alloy steel structural shapes been available. The light, non-symmetrical shapes warped easily from the quenching and usually wound up looking more like spaghetti than steel. Now U. S. Steel has solved the problem. The result is a new engineering material, available only from USS, that puts 2 to 3 times more strength to work but doesn't weigh an ounce more.



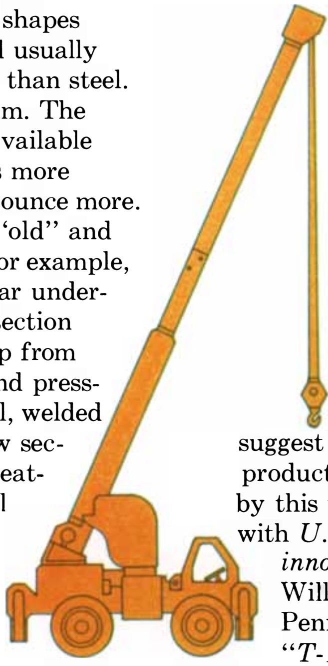
Look at these "old" and "new" sections, for example, used in subway car underframes. The old section (left) was built up from rolled channels and press-formed plate steel, welded together. The new section (right) is a heat-treated alloy steel shape only about half as heavy, and just as strong.

In a television broadcasting tower use of heat-treated alloy steel shapes cut steel weight 50% in the lower 187 feet. Used just for end posts in a fleet of railroad commuter cars, they trimmed 360 lbs. off each car. A crane boom manufacturer used heat-treated alloy shapes to cut his welding cost 50%. A heavy-duty truck builder used them to solve a severe impact problem, and eliminated a brake-forming operation in the bargain.



Savings as high as 30% aren't unusual when heat-treated alloy steel shapes replace sections that previously had to be fabricated by cutting and welding alloy steel plates. Many small rolled shapes can now be obtained for sections that were impractical to fabricate in the past.

Heat-treated alloy steel shapes culminate a U. S. Steel development program that began a decade ago with the introduction of USS "T-1" Steel, 3 times stronger than carbon steel and possessing remarkable toughness. "T-1" steel as well as other alloy steels that develop mechanical properties through heat treatment are now available from U. S. Steel as rolled shapes. Another USS innovation has led to the first quenched and tempered alloy steel bar flats. In their most unusual use so far, the thinner, stronger sections permit extra visibility through the bars of a gorilla cage.



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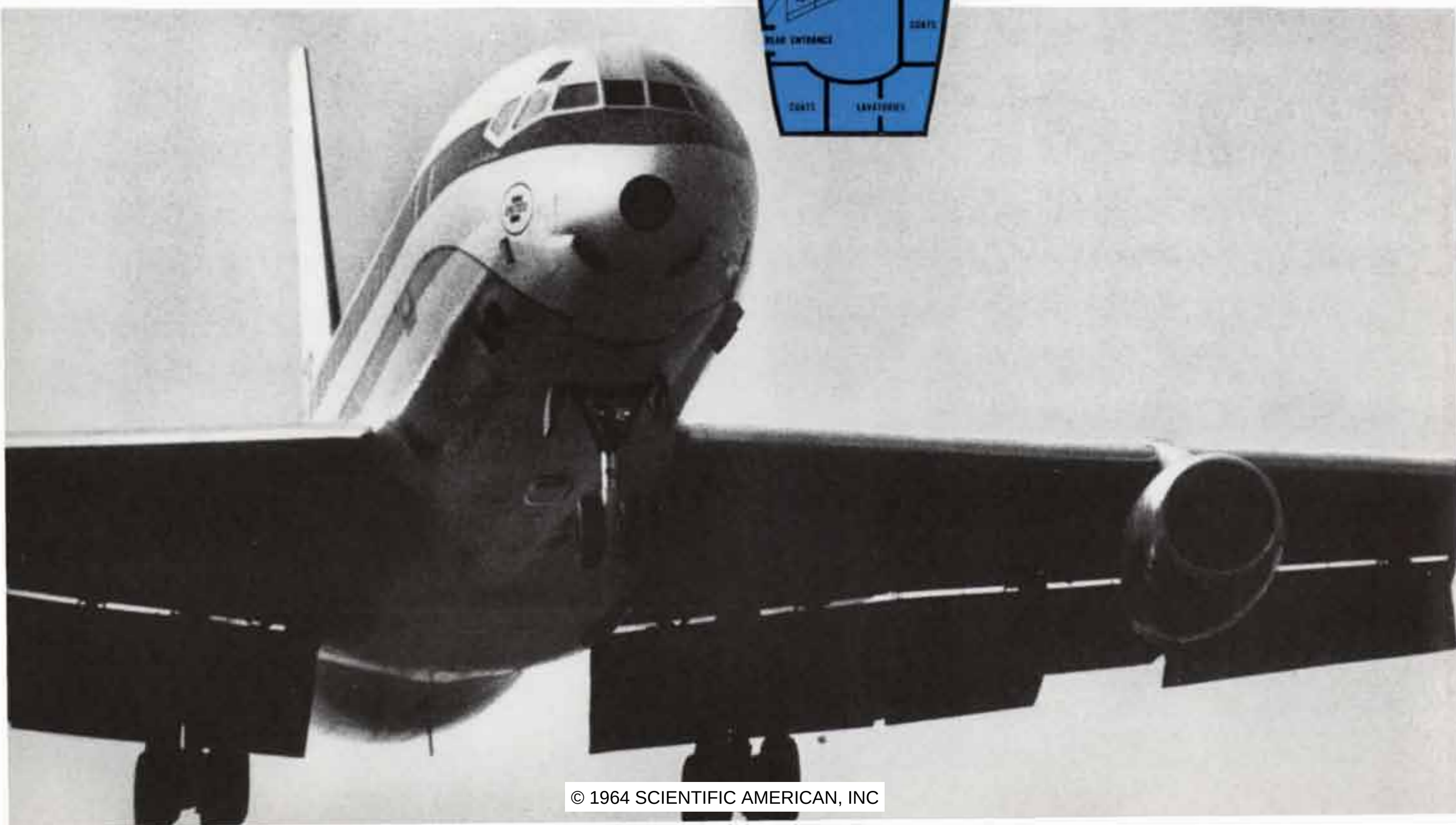

bars across his thighs, unless he could pay for easement, and the owner of the prison had bought it from the government because it was so profitable.

Hibbert is a bit grudging about admitting modern progress, although the contrasts in his descriptions show cruelty eventually greatly diminished. The need now as then is for understanding. Tout comprendre, c'est tout pardonner. Parole is the modern means for reducing the recurrence of crime, but there is seldom enough money to make it work well. Behind parole there needs to be understanding of the criminal mind. Hibbert is clear about the cult of criminality, and how the public has for centuries added to the vanity of wrongdoers, how a youth maims and injures, not for goods or money but for glory in the eyes of his gang. But Hibbert does not concentrate on the principal item in the criminal's psychology: his lack of social conscience, his egoistic disregard of the feelings of those who constitute his outgroup. Psychologists know a good deal now about how consciences are formed, but of course this knowledge alone is not enough to solve the problem. No punishment will provide a man with a new set of parents and a changed childhood. Nevertheless, something can be done by devising a belated substitution for a deleterious past, but that unfortunately requires a larger investment than society is yet willing to make.

THE YEAR OF THE GORILLA, by George B. Schaller. The University of Chicago Press ($5.95). Last year Schaller published *The Mountain Gorilla*, a field report of his observations during two years of an expedition in East and Central Africa. Although primarily a technical study of the behavior and ecology of the gorilla, it is a rewarding book for any reader curious about natural history. The present volume attends to the more personal side of Schaller's adventure—how he and his wife lived, his encounters with the native peoples, his moods and thoughts as he came to know the incredibly varied, teeming plant and animal life of Africa. Much of the material consists of descriptions, which, although skillfully handled, are often longwinded, and the aperture to the man himself is constricted. One of the revealing things about his personality is that he manages to say almost nothing about his wife, who spent the greater part of two years in a wretched hut in damp African hills (like Mrs. Tarzan) cooking, mending, patching, fighting rain, heat, insects and other hardships mostly waiting for Schaller to come home. A woman who can do that is more amazing than any creature Schaller found on his expedition.

The American Language, abridged by Raven I. McDavid, Jr. Alfred A. Knopf ($12.95). A revised and shortened version of this work, the fourth edition of which came out in 1936, followed by two supplements in 1945 and 1948. *The American Language,* as the editor says, is uniquely H. L. Mencken's and at the same time a work of serious scholarship "that underwent progressive modifications in successive versions." Even in the past 15 years, however, during which nothing was added to it (Mencken had his first paralytic stroke and never wrote again after the appearance of the second supplement), our language has changed, the great dictionaries such as the Oxford and Webster's have either been revised or are in the process of revision, new scholarly lexicons (for example *The Dictionary of Americanisms*) have been published and entirely new approaches have been developed in linguistics. These circumstances set many obstacles in the path of the editor. He was not content simply to abridge Mencken's text; on the other hand, he was not prepared fully to incorporate the linguistic advances and language changes of the past 15 years, since this would have required drastic alterations. He has compromised by condensing the three original volumes, "with updating where necessary and editorial commentary at critical points." Thus we have a generous measure of the old dish and a fresh seasoning of recent scholarship.

Bioastronautics, edited by Karl E. Schaefer. The Macmillan Company ($16). Arthur C. Clarke once suggested the image of the gravitational pit from which the man-carrying vehicle must be flung in order to attain the surface of an indefinite, extending smooth flat plane. It is hard enough to get out of the pit, but the astronaut's troubles really only begin when he glides around the Elysian fields. He is, after all, an organism designed for the solid earth and the ocean of air, and the physiological and psychological effects of moving about weightlessly in empty space are inordinately complex and largely unforeseeable. This book discusses some of the most important aspects of the biology of flight, including acceleration stresses, vibration and noise, heat tolerance and heat protection, radiation exposure, the biological effects of magnetic fields and

Norbert Wiener died this Spring.

One of the world's ranking mathematicians, Norbert Wiener was also a linguist, a philosopher, a literary scholar, a scientist with a conscience, a genial and witty man, a grandfather. Norbert Wiener died this Spring.

After a photograph by Y. W. Lee

Norbert Wiener's books include: THE FOURIER INTEGRAL AND SOME OF ITS APPLICATIONS (*Dover*); CYBERNETICS (*The M.I.T. Press*); EXTRAPOLATION, INTERPOLATION, AND SMOOTHING OF STATIONARY TIME SERIES WITH ENGINEERING APPLICATIONS (*The M.I.T. Press*; available in MIT paperbound under the title TIME SERIES); THE HUMAN USE OF HUMAN BEINGS (*Houghton Mifflin;* second revised edition, paperbound, *Doubleday*); EX-PRODIGY (*The M.I.T. Press*); I AM A MATHEMATICIAN (*Doubleday;* MIT paperbound, *The M.I.T. Press*); NONLINEAR PROBLEMS IN RANDOM THEORY (*The M.I.T. Press*); THE TEMPTER (*Random House*); GOD AND GOLEM, INC. (*The M.I.T. Press*); SELECTED PAPERS OF NORBERT WIENER (*The M.I.T. Press*).

THE M.I.T. PRESS, CAMBRIDGE, MASSACHUSETTS 02142 the disturbance of normal physiological rhythms. Two noteworthy articles are one on biological computers (W. Ross Ashby and Heinz von Foerster), a discussion of the computation principles of living organisms and their possible embodiment in electronic devices, and one on the experience of time dilation (Hermann von Schelling), which once more stirs up the twin paradox and argues that, however time may be affected by motion in the realm of physics, Einstein's *Eigenzeit* is "without a genuine meaning in physiology," and that the twins, when reunited, will be the same age.

CHAUNCEY WRIGHT AND THE FOUNDATIONS OF PRAGMATISM, by Edward H. Madden. University of Washington Press ($5). A biographical sketch and an examination of the philosophy of a 19th-century American who, by his writings and personal acquaintance, stimulated and markedly influenced a wide circle of thinkers, among them Charles Sanders Peirce, William James and Oliver Wendell Holmes, Jr. Wright was the "Socratic center" of this circle and is remembered alike for his keenness of mind, his honesty of inquiry and his independence.

NOBLE-GAS COMPOUNDS, edited by Herbert H. Hyman. The University of Chicago Press ($12.50). Beginning with the discovery in 1894 and 1895 by Lord Rayleigh and William Ramsay of the first of the so-called inert gases, argon, it was an accepted theory for nearly 70 years that none of these indolent substances would deign to form stable chemical compounds. In June, 1962, however, the theory was murdered by facts: the first true compound of the "inert" gas xenon was produced at the University of British Columbia by Neil Bartlett and N. K. Jha. Today, some two years later, it is known not only that xenon has a chemistry but also that the noble gases have a much richer and more complicated chemistry than was ever dreamed of. This volume consists of papers presented at a conference held at the Argonne National Laboratory in September, 1962, and touches on many different aspects of this exciting and promising new field.

PUTZGER: HISTORISCHER WELTATLAS, by F. W. Putzger. Velhagen & Klasing ($4). THE PENGUIN ATLAS OF MEDIEVAL HISTORY, by Colin McEvedy. Penguin Books, Inc. ($2.45). Putzger's is the 84th edition (the first was published in 1877) of this grandfather of all mod-

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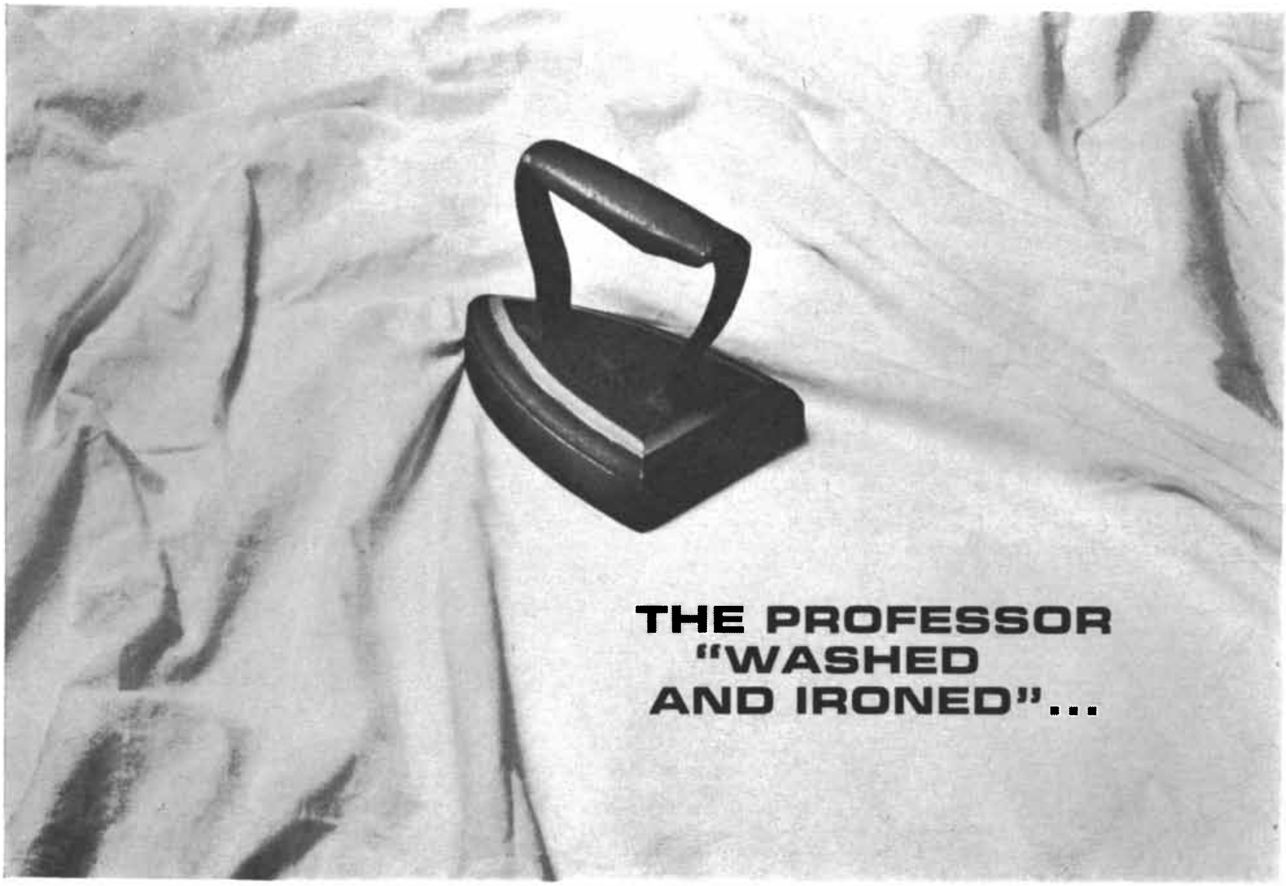
GREAT MOMENTS AT CHICAGO



In 1909 a young professor at the University of Chicago worked late, missing a dinner party. One of the guests was baffled to hear the explanation: "he had washed and ironed for an hour and a half and had to finish the job."

Robert A. Millikan was indeed hard at work, *watching an ion* – an ionized droplet of oil suspended in air between the opposing forces of gravity. When the "job" was finally finished – hundreds of hours of patient observation later – the young professor had demonstrated for the first time the unitary nature of electricity, a discovery which later brought him the Nobel Prize.* The University of Chicago provides a highly charged atmosphere for scientific research. And the University of Chicago Press is ever watchful for scientific developments around the world. The result is books like these:

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Snow Crystals, by W. A. Bentley and W. J. Humphreys. Dover Publications, Inc. ($3). An unaltered and unabridged republication of this collection of some 2,000 snow-crystal photographs made by Bentley in Vermont over a period of more than half a century. Humphreys, a meteorologist, explains how the pictures were made and the method of classification according to crystallographic peculiarities. A remarkable volume.

From Low-Speed Aerodynamics to Astronautics, by Theodore von Kármán. Pergamon Press ($3.75). The late Theodore von Kármán gave these lectures at the University of Maryland in April, 1961, just before his 80th birthday. They constitute a brief, elegant history of the development of aerodynamics from the 19th century to contemporary advances in magnetofluid dynamics. Von Kármán maintained that he was addressing himself to the nonspecialist as well as to the specialist, but for all the clarity and originality of his presentation this is very much a book for those who know their stuff. Illustrations.

Symbol, Status, and Personality, by S. I. Hayakawa. Harcourt, Brace & World, Inc. ($4.50). A group of urbane essays on different aspects of the use and misuse of language. Hayakawa is a thoughtful student of semantics with a sober sense of social responsibility; even though this is merely a collection of magazine pieces and no attempt has been made to give the book structure, one can profit from its trenchant and often witty observations.

Evidence for Gravitational Theories, directed by C. Møller. Academic Press ($10). This book, one of the courses of the International School of Physics "Enrico Fermi," consists of lectures given at Varenna on Lake Como in the summer of 1961. Among the participants were R. H. Dicke (Mach's principle and equivalence), Fred Hoyle (cosmological tests of gravitational theories), J. Weber (theory of methods for the measurement and production of gravitational waves), B. Bertotti (the theory of measurement in general relativity) and H. Bondi (gravitational waves).

THE CHAMBERED TOMBS OF SCOTLAND, VOLUME I, by Audrey Shore Henshall. Aldine Publishing Company ($20). The first volume of a corpus of the Scottish prehistoric chamber tombs, consisting of minutely detailed descriptions, abundantly supported by maps, plates, ground plans and other illustrations. For each of the five groups of tombs the text deals with distribution, siting, typology, ritual and contents. This is an invaluable scholarly publication assembling the basic elements for British prehistory.

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EVOLUTION: THE MODERN SYNTHESIS, by Julian Huxley. John Wiley & Sons, Inc. ($2.65). A paperback reprint of Huxley's most valuable work, a major contribution to the theory of evolution.

MAGNETISM, edited by George T. Rado and Harry Suhl. Academic Press ($37). Two volumes have appeared so far of this comprehensive treatise that affords a summary of present knowledge of magnetically ordered materials. Volume I covers magnetic ions in insulators; Volume III, spin arrangements and crystal structure, domains and micromagnetics.

ORIGIN OF THE SOLAR SYSTEM, edited by Robert Jastrow and A. G. W. Cameron. Academic Press ($8). Proceedings of a conference held in 1962 at the Goddard Institute for Space Studies, including articles on the history of theories of the origins of the solar system (D. ter Haar and A. G. W. Cameron),

star formation (Lyman Spitzer, Jr.), formation of the planets (Fred Hoyle), formation of the solar nebula (Cameron), meteorites and the early history of the solar system (Edward Anders). Illustrations.

THE LIVER: MORPHOLOGY, BIOCHEMISTRY, PHYSIOLOGY, VOLUME I, edited by Charles Rouiller. Academic Press ($26). The first installment of a two-volume treatise on present knowledge of the liver as based on morphological, biochemical and physiological studies. Illustrations and an extensive bibliography.

FOUNDATIONS OF EXPERIMENTAL EMBRYOLOGY, edited by Benjamin H. Willier and Jane M. Oppenheimer. Prentice-Hall, Inc. ($7.95). A collection of original papers by such noted pioneers as W. Roux, Hans Driesch, Edmund B. Wilson, Theodor Boveri, Ross G. Harrison, Otto Warburg, Frank R. Lillie, C. M. Child and others. Illustrations.

ELSEVIER'S MEDICAL DICTIONARY, compiled and arranged by A. Sliosberg. American Elsevier Publishing Company, Inc. ($45). A five-language medical dictionary—English/American, French, Italian, Spanish and German—arranged on an English alphabetical base.

TRISTES TROPIQUES, by Claude Lévi-Strauss. Atheneum ($2.45). An inexpensive reissue of an anthropological study of primitive societies in Brazil by a leading student and fieldworker. Many illustrations.

A HANDBOOK OF PRACTICAL AMATEUR ASTRONOMY, edited by Patrick Moore. W. W. Norton & Co., Inc. ($5.95). An observational guide for the serious amateur astronomer, with instructions on everything from the selection of telescopic equipment to astronomical calculations. Illustrated.

METHOD OF STATISTICAL TESTING, by Yu. A. Shreider. American Elsevier Publishing Company, Inc. ($15). A translation from the Russian of papers on different aspects of the Monte Carlo method.

ESSAYS ON THE THEORY OF NUMBERS, by Richard Dedekind. Dover Publications, Inc. ($1). A reprint of Dedekind's two classic essays, the first "Continuity and Irrational Numbers," in which he sets forth his famous concept of the cut, which defines a real number; the second "The Nature and Meaning of

Numbers," which lays a logical foundation for whole numbers. Paperback.

The Mitochondrion, by Albert L. Lehninger. W. A. Benjamin, Inc. ($9). This book summarizes all the cytological, biochemical, physiological and structural information on the mitochondrion, which generates the power of an aerobic cell. Electron micrographs, tables of data, drawings and bibliographies.

History of Hydraulics, by Hunter Rouse and Simon Ince. Dover Publications, Inc. ($2). A paperback reprint of a scholarly, well-written general history of hydraulics that has been kept free enough of mathematics to be accessible to almost any nonspecialist interested in the history of science. Many illustrations.

The Lower Metazoa, edited by Ellsworth C. Dougherty and others. University of California Press ($17.50). A collection of some 30 monographs on the comparative biology and phylogeny of the lower Metazoa.

A Million and One Nights, by Terry Ramsaye. Simon and Schuster, Inc. ($3.95). A paperback reissue of Ramsaye's engrossing history of the motion picture from its beginnings through 1925. Fully illustrated and attractively told by a journalist who devoted a good part of his life to the motion picture industry and probably knew more about the period he covers than anyone else, this is a delightful book.

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The Chemistry of Thinking, by George Humphrey and R. V. Coxon. Charles C Thomas, Publisher ($6.75). An approach to cerebral function based on the findings of experimental psychologists and biochemists, much of the work being devoted to a summary of evidence regarding chemical reactions that occur in nervous tissue.

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SATELLIT

Today, the big word in satellites is "synchronous." The first was NASA's Syncom—just chalking up a full year of failure-free service. Its sister—"Public Satellite #1"—will bring Europe close as your phone, let you see even the longest overseas TV programs "live" and uninterrupted.

There was more than a little skepticism in 1959 when Hughes engineers began work on a synchronous satellite. The problem was to boost a satellite to a precise altitude of 22,300 miles and position it so it would travel with the earth's rotation – in effect "stand still" over a point on earth.

Each such "star" would be in constant sight of a third of the earth. Since it remains "parked" it could beam signals to and from the inexpensive type of ground stations which have stationary antennas. These will be affordable probably by all nations with the need. The net result: uninterrupted, 24-hour intercontinental communications. Improved telephone, wireless, wirephoto and TV communications would not be the only benefits. Other uses might include: instant inventory control on global basis, truck to truck military field communications, aircraft flight control, navigation and meteorology.

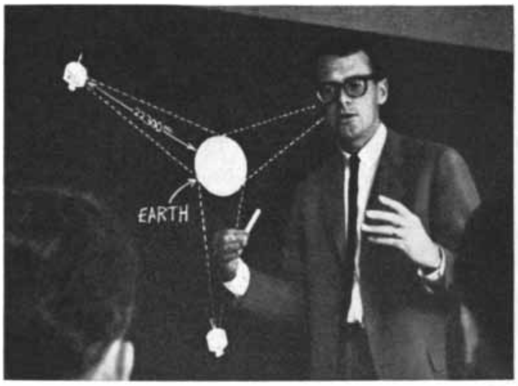
Thus, a great deal rode on Syncom when it was launched on July 26, 1963. But in the year since, skepticism has all but evaporated. Syncom has transmitted the voices of thousands of people and other kinds of signals for over 2,500 hours. It has set a record of more than twice the communication time logged by other satellite types. An almost incredible series of space feats occurred, too. After arriving at launch altitude over the Indian Ocean, Syncom was "walked" halfway around the world to a six-month stay over Brazil and recently was taken for another "walk" to its present position over the Pacific.

These achievements helped bring the first truly global communications system closer to reality when the Communications Satellite Corporation chose a Syncom-type satellite to be "Public Satellite #1." Any American will be able to make a call via "Early Bird's" facilities. It will almost double present trans-Atlantic telephone capacity; while being instantly ready to bring you historic TV programs from overseas.

Hughes is also active in other areas of the technology. One of these – ground stations–can be available when needed. Hughes is now under contract to NASA to further extend man's knowledge of space with Advanced Technological

UNRETOUCHED PHOTO OF PRESIDENT KENNEDY SENT VIA SYNCOM LAST SUMMER DURING THE HISTORIC EXCHANGE OF PHOTOS WITH PRESIDENT AZIKIWE OF NIGERIA.

"PUBLIC SATELLITE #1" (RIGHT) IS SCHEDULED FOR LAUNCH BY THE COMMUNICATIONS SATELLITE CORPORATION EARLY IN 1965. "EARLY BIRD" WILL LET YOU DIAL EUROPE DIRECT. SEE OVERSEAS TV PROGRAMS "LIVE" AND UNINTERRUPTED.



SIMPLICITY OF THE SYNCHRONOUS SATELLITE CONCEPT IS ILLUSTRATED. POSITIONED AT AN ALTITUDE OF 22,300 MILES, ONE OPERATING SYNCOM-TYPE SATELLITE CAN COVER A THIRD OF THE EARTH.

SYNCOM UNDERGOING CHECKOUT (RIGHT) BEFORE LAUNCH A YEAR AGO. NOT ONE OF ITS 2,000 PARTS HAS FAILED – RELIABILITY WHICH HAS ENABLED IT TO BREAK COMMUNICATIONS SATELLITE RECORDS.

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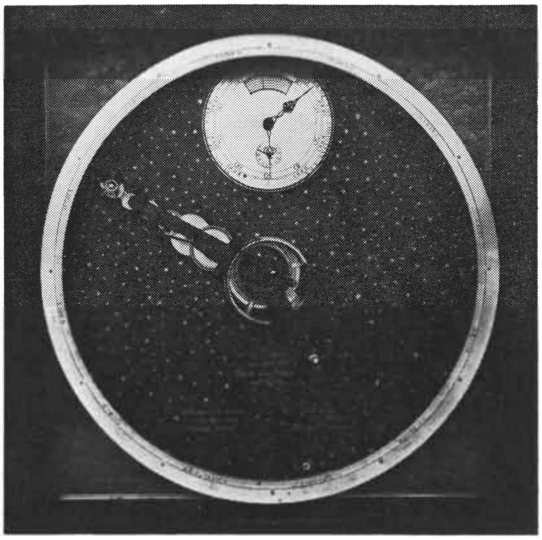

Great things have happened since 1771 when David Rittenhouse's orrery* took its place in Nassau Hall as "the crowning glory of the College's philosophical apparatus." For one thing, Science — then a fledgling among mature disciplines—has flourished. Today, great University communities prepare special environments for advanced study and for explorations into the unknown. Discovery feeds upon discovery.

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*Invented to demonstrate Newtonian principles of astronomy, the Rittenhouse orrery at Princeton was damaged during the Revolutionary War, repaired in the early 19th Century, and restored in 1952-53. It is now in the Firestone Library at Princeton. Brooke Hindle's biography, *David Rittenhouse*, will be published in October.

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it is by standing on the shoulders of giants."
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